MATHEMATICS
Analysis and Approaches - HL

ANSWERS

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6th Edition

For use with the I.B. Diploma Programme

Exercise A.5.1

1	15						
2	a	25	b	625			
3	a	24	b	256			
4	a	24	b	48			
5	15						
6	270						
7	120						
8	336						
9	60						
10	a	362880	b	80640	c	1728	
11	20						
12	a	10!	b	2×8!	c i 2×9!	ii 8×9!	
13	34650						
14	4200						
15	4						

Exercise A.5.2

1	792			
2	a	1140	b	171
3	1050			
4	70			
5	2688			
6	a	210	b	420
7	889			
9	24000			
10	8			
11	155			
12	5			

Exercise A.5.3

1	a	120	b	325
2	5040			
3	a	144	b	1440
4	a	720	b	240
5	11760			
6	7056; 4606			
7	a	840	b	1680
8	190			
9	10080			
10	226800			
11	a 71 b 315 c 665			
13	${}^{n}C_2$			

14 $^{n}C_4$

15 b 92

16 252

17 a 1287 b 560

18 256

19 288

20 a 10 080 b 30 240 c 14 400

21 10 080, 1080

22 3 528 000

23 720; 240

24 103 680

25 a 12 b 128

26 2880

27 a 30 030 b 37 310

28 77 055

29 a 48 b 72

Exercise A.5.4

1. $1-2x+3x^2-4x^3+5x^4$ a 0.5265 b Not even correct to 1 s.f.

2. $1-0.5x+0.375x^2-0.3125x^3+0.273438x^4-0.246094x^5$ a 0.9951

   b High accuracy > 6 s.f.

3. a $0.00295474x^6$ b $-0.0205078x^6$

   c $64x^6$ d $-1.3125x^6$

4. a $1.-1.5x+3.375x^2-8.4375x^3+22.1484x^4-59.8008x^5+164.452x^6$

   b $1.-2.1x-0.945x^2-1.2285x^3-2.11916x^4-4.19594x^5-9.02127x^6$

   c $1+\frac{2}{3}x-\frac{4}{9}x^2+\frac{40}{81}x^3-\frac{160}{243}x^4+\frac{704}{729}x^5-\frac{9856}{6561}x^6$

   d $2+x-\frac{1}{4}x^2+\frac{1}{8}x^3-\frac{5}{64}x^4+\frac{7}{128}x^5-\frac{21}{512}x^6$

5. a $3+6x+9x^2+12x^3+15x^4+18x^5+21x^6$

   b 4.6871

   c 3.7037

6. $-3.5x^5$

7. a $4+x=4\left(1+\frac{1}{4}x\right)$ b $\sqrt{4+x}=\sqrt{4\left(1+\frac{1}{4}x\right)}=2\sqrt{1+\frac{1}{4}x}$

c $2.+0.25x-0.015625x^2+0.00195313x^3-...$ d 2.0249

8. a $1+15x+150x^2$ b 437 500

c 7.5 growing. d 4.167 growing, but not so rapidly.

e 3.33 growth slowing. f No. It converges too slowly.

9. $1.-0.25x+0.15625x^2-0.117188x^3+0.0952148x^4-0.0809326x^5+0.070816x^6$

a 0.976454 b < 0.0000001 c <0.00001%

10. 0.4774

11. 1.414

12. $\sqrt{1-x}\approx 1.-0.5x-0.125x^2-0.0625x^3-0.0390625x^4$ and $\frac{1}{(1+x)^2}\approx 1-2x+3x^2-4x^3+5x^4$

a $1-2.5x+3.875x^2-5.3125x^3+6.71094x^4$

b 0.784109 c –.00007 d ~0.007%

13. a 1.025 b 2.012 c 0.9706

d 1.094 e 4.921 f 14.21

Exercise A.6.1

1. a $\frac{2}{x+2}+\frac{3}{x-3}$ b $\frac{1}{x+4}+\frac{2}{x-7}$ c $\frac{3}{x+1}-\frac{4}{x-3}$

   d $\frac{1}{2x+1}-\frac{2}{x-3}$ e $\frac{1}{2x+3}+\frac{2}{3x-1}$

2. a $\frac{2}{(x-1)^2}+\frac{3}{x-1}$ b $\frac{4}{(x-3)^2}-\frac{2}{x-3}$ c $\frac{1}{(x-1)^3}+\frac{4}{(x-1)^2}-\frac{2}{x-1}$

   d $\frac{1}{(x-2)^3}+\frac{1}{(x-2)^2}+\frac{2}{x-2}$ e $\frac{3}{(2x+1)^3}+\frac{1}{(2x+1)^2}+\frac{4}{2x+1}$

3. a $\frac{1}{x+2}+\frac{x+1}{x^2+2}$ b $\frac{1}{x+2}+\frac{2}{x^2+x+7}$ c $\frac{3}{2x+1}-\frac{2}{x^2+3x+1}$

   d $\frac{1}{x+2}+\frac{3}{x^2+7}$ e $\frac{4}{x+5}+\frac{3x-2}{x^2+5}$

4. a $3-\frac{2}{x+1}$ b $3+\frac{x}{x^2+2}$ c $2x-1+\frac{x}{x^2+5}$

   d $x-2+\frac{3}{(x-1)^2}$

5. a $\frac{1}{x-1}+\frac{-x}{x^2+1}$ b $\frac{0.5x}{x^2+3}+\frac{-0.5x}{x^2+5}$ c $1-\frac{3x+3}{x^3-1}$

   d $\frac{2}{x-1}+\frac{-2x-1}{x^2+x+1}$ e $x+\frac{4x}{x^2-4}$ f

Exercise A.7.1

1 a i 2 ii -3 iii 6 iv 0 v $\frac{3}{2}$ vi $\frac{1}{3}$

b i 2 ii $\sqrt{2}$ iii -5 iv $-\frac{2}{5}$ v $\frac{1}{2}$ vi -1

c i $2-2i$ ii $-3-\sqrt{2}i$ iii $6+5i$

iv $\frac{2}{5}i$ v $\frac{3}{2}-\frac{1}{2}i$ vi $\frac{1}{3}+i$

2 a $7+i$ b $1-3i$ c $15-8i$
d $-1-8i$ e $10+11i$ f $-2+3i$

3 a $-1+3i$ b $5-i$ c $-4+3i$
d $6i$ e $-4+7i$ f $-2+3i$

4 a $\frac{1}{2}(1+i)$ b $-\frac{1}{2}(5+i)$ c $-1-2i$

d $\frac{1}{2}i$ e 1 f $-\frac{1}{13}(5+i)$

5 a $14+8i$ b $-2-2i$ c $-2\sqrt{2}-i$

d $\frac{1}{5}(2+i)$ e $2-i$ f $\frac{1}{5}(1+3i)$

6 a $\frac{1}{2}$ b $\frac{1}{2}(3+\sqrt{2})$ c $3+\sqrt{2}$

7 a $x = 4, y = -\frac{1}{2}$ b $x = -5, y = 12$ c $x = 0, y = 5$

8 a i $1, i, -1, -i, 1, i$ ii $-i, -1, i, 1, -i$

b i -1 ii $-i$ iii -1 iv -1

9 $x = -\frac{120}{29}, y = \frac{39}{29}$

12 a $x = 0$ or $y = 0$ or both b $x^2 - y^2 = 1$

13 a $3-i$ b $2-i$

14 a $4i$ b -4 c $-i$

15 a $x = 13, y = 4$ b $x = 4, y = \frac{4}{3}$

16 1

17 $-\frac{1}{3}(1+2\sqrt{2}i)$

18 $(u, v) = \left(\frac{1}{2}(\sqrt{2}+2), \frac{1}{2}\sqrt{2}\right), \left(\frac{1}{2}(2-\sqrt{2}), -\frac{1}{2}\sqrt{2}\right)$

19 a $-\frac{7}{2}$ b $-\frac{1}{5}$

21 $\pm\frac{\sqrt{2}(1+i)}{2}$

22 a $\cos(\theta+\alpha)+i\sin(\theta+\alpha)$ b $\cos(\theta-\alpha)+i\sin(\theta-\alpha)$

c $r_1r_2(\cos(\theta+\alpha)+i\sin(\theta+\alpha))$

d $x^2-2x\cos(\theta)+1$ e $x^2+2x\sin(\alpha)+1$

24 a $3+i$ b 325 c $(x^2+y^2)^2$

25 $z = 4, b = -4$

26 a $\cos(\theta)+i\sin(\theta)$ b $\cos(4\theta)+i\sin(4\theta)$

27 a $\begin{bmatrix}-\alpha^2 & 0\\ 0 & -\beta^2\end{bmatrix}$ b $\begin{bmatrix}\alpha^4 & 0\\ 0 & \beta^4\end{bmatrix}$

c $\begin{bmatrix}-\frac{i}{\alpha} & 0\\ 0 & \frac{i}{\beta}\end{bmatrix}$ d $\begin{bmatrix}\alpha^{4n} & 0\\ 0 & \beta^{4n}\end{bmatrix}$

28 a $-\sin(\theta)+i\cos(\theta)$ d $\cos(\theta)-i\sin(\theta)$

Exercise A.7.2

1. Show the following complex numbers on an Argand diagram:

g $\frac{1}{2i}$ h $\frac{2}{1+i}$

3. If $z_1 = 1+2i$ and $z_2 = 1+i$, show each of the following on an Argand diagram:

g $\frac{z_1}{z_2}$ h $\frac{z_2}{z_1}$

4. Find the modulus and argument of:

d $3i$ e $-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ f $\frac{1}{\sqrt{2}}(i+1)$

g 6 h $\left(1-\frac{1}{2}i\right)^2$

14. Determine the modulus and argument of each of the complex numbers:

a $3-4i$ b $\dfrac{2}{1+i}$ c $\dfrac{1-i}{1+i}$

15. If $z = 1+i$ find $Arg(z)$. hence, find $Arg\left(\dfrac{1}{z^4}\right)$.

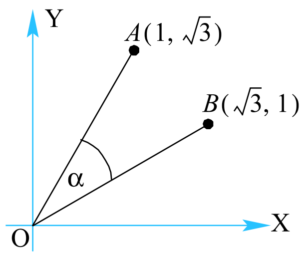
16. Determine the modulus and argument of each of the complex numbers:

a $\cos\theta + i\sin\theta$ b $\sin\theta + i\cos\theta$ c $\cos\theta - i\sin\theta$

17. Find the modulus and argument of:

a $1 + i\tan\alpha$ b $\tan\alpha - i$ c $1 + \cos\theta + i\sin\theta$

18. i Express $\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}$ in the form $u+vi$.



ii Let α be the angle as shown in the diagram. Use part i to find α, clearly explaining your reason(s).

Hence, find $Arg(z)$ where $z = \left(\dfrac{1+\sqrt{3}i}{\sqrt{3}+i}\right)^7$.

19. Find:

i the modulus

ii the principal argument of the complex number $1-\cos\theta - i\sin\theta$.

On an Argand diagram, for the case $0<\theta<\pi$, interpret geometrically the relationship:

$$1-\cos\theta - i\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta-\pi}{2}\right) + i\sin\left(\frac{\theta-\pi}{2}\right)\right).$$

20. If $z = \cos\theta + i\sin\theta$, prove:

a $\dfrac{2}{1+z} = 1 - i\tan\left(\dfrac{\theta}{2}\right)$.

b $\dfrac{1+z}{1-z} = i\cot\left(\dfrac{\theta}{2}\right)$.

Exercise A.7.3

1 a $\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ b $\sqrt{2}\text{cis}\left(\frac{3\pi}{4}\right)$ c $\sqrt{2}\text{cis}\left(-\frac{3\pi}{4}\right)$

2 a $2\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$ b $2\text{cis}\left(\frac{\pi}{6}\right)$ c $4\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$

d $5\text{cis}(53°7')$ e $\sqrt{5}\text{cis}(153°26')$ f $\sqrt{13}\text{cis}(-123°41')$

g $2\text{cis}\left(\frac{5\pi}{6}\right)$ h $\text{cis}\left(-\frac{\pi}{3}\right)$ i $\sqrt{10}\text{cis}(-18°26')$

3 a $2i$ b $\frac{3\sqrt{3}}{2}+\frac{3}{2}i$ c $1-i$

d $-5i$ e $-4+4\sqrt{3}i$ f $\frac{1}{6}(\sqrt{2}+\sqrt{6}i)$

4 a $\sqrt{\frac{5}{3}}$ b 1 c 0

5 a $1-\sqrt{3}i$ b $1-i$ c $(1-\sqrt{3})+(1+\sqrt{3})i$

7 a $\sqrt{2}$ b 2 c $2\sqrt{2}$

d $\frac{\pi}{4}$ e $\frac{2\pi}{3}$ f $\frac{11\pi}{12}$

Exercise A.7.4

1 a $-4(1+i)$ b -4 c $-16+16i$

d $-8-8\sqrt{3}i$ e $-16\sqrt{3}-16i$ f $-117-44i$

2 a $\frac{1}{8}(-1+i)$ b $-\frac{1}{4}$ c $-\frac{1}{32}(1+i)$

d $\frac{1}{32}(-1+\sqrt{3}i)$ e $\frac{1}{64}(-\sqrt{3}+i)$ f $\frac{1}{15625}(-117+44i)$

3 a $-8i$ b $\frac{81}{2}(-1+\sqrt{3}i)$ c $\frac{1}{2}i$

d $-\frac{1}{125}i$ e $-\frac{1}{16}(1+\sqrt{3}i)$ f $-\frac{2}{81}(1+\sqrt{3}i)$

4 a $128(1-i)$ b $4\sqrt{3}-4i$ c $-32i$

d 256 e $\frac{11753}{625}-\frac{10296}{625}i$ f $-2i$

5 b $i-1$ ii -1 iii i

6 a $-i$ b $6\sqrt{2}(1+i)$ c $-\sqrt{2-\sqrt{2}}+\sqrt{2+\sqrt{2}}i$

7 a $\frac{\sqrt{2}}{2}(1+i)$; $\frac{1}{2}(1+\sqrt{3}i)$ $\frac{\sqrt{2}}{4}((1-\sqrt{3})+(1+\sqrt{3})i)$

b i $\frac{\sqrt{2}}{4}(1+\sqrt{3})$ ii $\frac{\sqrt{2}}{4}(1-\sqrt{3})$

c $3(\text{cis}(-\theta))^3$

16 $(\cos 2\theta + \cos 2\alpha)(\cos(\theta - \alpha) - i\sin(\theta - \alpha))$ [or $2\cos(\alpha - \theta)$]

18 a $\text{cosec}\,\theta$ b $\theta - \frac{\pi}{2}$

Exercise A.7.5

1 a $-\frac{3}{2} \pm \frac{3\sqrt{3}}{2}i, 3$ b $\pm\frac{3\sqrt{3}}{2} + \frac{3}{2}i, -3i$ c $2i, \pm\sqrt{3} - i$

d $-\sqrt{2} - \sqrt{2}i, -\sqrt{2} + \sqrt{2}i, \sqrt{2} - \sqrt{2}i, \sqrt{2} + \sqrt{2}i$

e $\frac{3}{2}(\sqrt{2-\sqrt{2}} - \sqrt{2+\sqrt{2}}i), -\frac{3}{2}(\sqrt{2-\sqrt{2}} - \sqrt{2+\sqrt{2}}i), \frac{3}{2}(\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}i)$ $-\frac{3}{2}(\sqrt{2+\sqrt{2}} + \sqrt{2-\sqrt{2}}i)$

f $\pm 2, \sqrt{3} \pm i, -\sqrt{3} \pm i$

2 $1 \pm i, -1 \pm i$; $(z-1-i)(z-1+i)(z+1-i)(z+1+i)$

3 a $\pm\frac{1}{\sqrt{2}}(1+i)$ b $2+i, -2-i$ c $\pm\frac{\sqrt{2}}{2}(1+\sqrt{3}i)$

4 a $\sqrt[6]{2}\,\text{cis}\left(-\frac{\pi}{12}\right), \sqrt[6]{2}\,\text{cis}\left(\frac{7\pi}{12}\right), \sqrt[6]{2}\,\text{cis}\left(-\frac{9\pi}{12}\right)$

b $\sqrt[3]{2}\,\text{cis}\left(\frac{2\pi}{9}\right), \sqrt[3]{2}\,\text{cis}\left(\frac{8\pi}{9}\right), \sqrt[3]{2}\,\text{cis}\left(-\frac{4\pi}{9}\right)$

c $\text{cis}\left(\frac{\pi}{6}\right), \text{cis}\left(\frac{5\pi}{6}\right), \text{cis}\left(-\frac{\pi}{2}\right)$

5 a $\sqrt[8]{2}\,\text{cis}\left(\frac{\pi}{16}\right), \sqrt[8]{2}\,\text{cis}\left(\frac{9\pi}{16}\right), \sqrt[8]{2}\,\text{cis}\left(-\frac{15\pi}{16}\right), \sqrt[8]{2}\,\text{cis}\left(-\frac{7\pi}{16}\right)$

b $\text{cis}\left(\frac{\pi}{8}\right), \text{cis}\left(\frac{5\pi}{8}\right), \text{cis}\left(\frac{9\pi}{8}\right), \text{cis}\left(\frac{13\pi}{8}\right)$ c $i, \pm\frac{\sqrt{3}}{2} - \frac{1}{2}i$

d $2\text{cis}\left(-\frac{\pi}{12}\right), 2\text{cis}\left(\frac{5\pi}{12}\right), 2\text{cis}\left(\frac{11\pi}{12}\right), 2\text{cis}\left(-\frac{7\pi}{12}\right)$ e $2(\pm\sqrt{3} + 1i), -4i$

f $\pm\frac{1}{2}((\sqrt{3}+1) + (\sqrt{3}-1)i)$

6 a $1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

7 $-2, 1 \pm \sqrt{3}i$

Exercise A.7.6

1 a $(x-3+i)(x-3-i)$ b $(x+2+3i)(x+2-3i)$
c $(x-1+i)(x-1-i)$ d $(z+2+i)(z+2-i)$

e $\left(z-\frac{(3+\sqrt{7}i)}{2}\right)\left(z-\frac{(3+-\sqrt{7}i)}{2}\right)$ f $(z+5+\sqrt{5}i)(z+5-\sqrt{5}i)$

g $4\left(w+\frac{1-4i}{2}\right)\left(w+\frac{1+4i}{2}\right)$ h $3(w-1+i)(w-1-i)$

i $-2\left(w-2-\frac{\sqrt{6}i}{2}\right)\left(w-2+\frac{\sqrt{6}i}{2}\right)$

2 a $-2\pm 2i$ b $\frac{1\pm\sqrt{11}i}{2}$ c $\frac{3\pm\sqrt{3}i}{6}$ d $\frac{-5\pm\sqrt{7}i}{4}$

e $-5\pm 2i$ f $\pm 4i$ g $-6\pm 2i$ h $-3\pm i$
i $\pm\frac{5}{3}i$

3 a $\pm 2, \pm i$ b $\pm 3, \pm i$ c $\pm 3, \pm 2i$

4 a $(z-5i)(z+5i)$ b $(z-7i)(z+7i)$
c $(z+2+i)(z+2-i)$ d $(z+3+\sqrt{2}i)(z+3-\sqrt{2}i)$
e $(z-2i)(z+2i)(z-\sqrt{2})(z+\sqrt{2})$
f $(z-\sqrt{2}i)(z+\sqrt{2}i)(z-\sqrt{3})(z+\sqrt{3})$

Exercise A.7.7

1 a $(z+2)(z+i)(z-i)$ b $(z-9)(z+i)(z-i)$
c $(z-2)(z+\sqrt{2}i)(z-\sqrt{2}i)$

2 a $(w+1-\sqrt{5}i)(w+1+\sqrt{5}i)(w-2)$
b $(z-1)(z-2+i)(z-2-i)$
c $(z-1)(z+1+i)(z+1-i)$
d $(x+2)(x-2)(x+i)(x-i)$
e $(w+2)(w-1+i)(w-1-i)$
f $(z+5)(z-5)(z+5i)(z-5i)$

3 a $1, 3\pm 4i$ b $2, 3\pm 2i$ c $-2, 3, 1\pm i$
d $\frac{1}{2}, -1\pm i$ e $-\frac{5}{3}, \frac{3}{2}, 1\pm\sqrt{2}i$ f $-1, -3\pm i$

4 $\frac{1}{3}, \frac{-1\pm\sqrt{3}i}{2}$

5 $-\frac{1}{2}, 1\pm 2i$

6 $(z-3)(z-2+3i)(z-2-3i)$

7 $1\pm 2i, \frac{-1\pm\sqrt{11}i}{2}$

8 a $(2z-1)(z+i)(z-i)$ b $(z+\sqrt{3})(z-\sqrt{3})(z+2i)(z-2i)$

9 $2 \pm i, -1$

10 $2 \pm 3i, -\frac{13}{4}$

11 $2 \pm i, \frac{1}{2}$

12 $(z-2)(z-4+i)(z-4-i)$

13 $(z-2)(z-1+i)(z-1-i)$

14 a $-2, 1, \frac{-1 \pm \sqrt{3}i}{2}, 1 \pm \sqrt{3}i$ b $1, 2, \frac{1 \pm \sqrt{3}i}{2}, -1 \pm \sqrt{3}i$

c $\pm\sqrt{3}, \pm i$ d $\pm\sqrt{5}, \pm \mathrm{i}$

15 a $x^3 - 7x^2 + 17x - 15 = 0$ b $x^4 - 5x^3 + 10x^2 - 10x + 4 = 0$

c $x^3 - 5x^2 + 10x - 12 = 0$ d $x^4 + 2x^3 + 2x^2 - 2x + 21 = 0$

16 $-1 - i\sqrt{3}$, $1 \pm \sqrt{5}$

17 $2 \pm \mathrm{i}, \frac{1}{2}(-3 \pm \sqrt{5})$

Exercise A.9.1

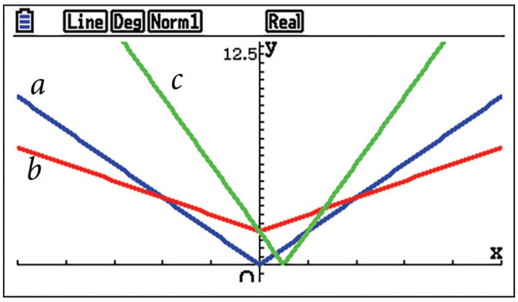
1	a	$x = 1, y = 2$	b	$x = 3, y = 5$	c	$x = -1, y = 2$
	d	$x = 0, y = 1$	e	$x = -2, y = -3$	f	$x = -5, y = 1$
2	a	$x = \frac{13}{11}, y = \frac{17}{11}$	b	$x = \frac{9}{14}, y = \frac{3}{14}$	c	$x = 0, y = 0$
	d	$x = \frac{4}{17}, y = -\frac{22}{17}$	e	$x = -\frac{16}{7}, y = \frac{78}{7}$	f	$x = \frac{5}{42}, y = -\frac{3}{28}$
3	a	-3	b	-5	c	-1.5
4	a	$m = 2, a = 8$	b	$m = 10, a = 24$	c	$m = -6, a = 9$.
5	a	$x = 1, y = a - b$	b	$x = -1, y = a + b$	c	$x = \frac{1}{a}, y = 0$
	d	$x = b, y = 0$	e	$x = \frac{a-b}{a+b}, y = \frac{a-b}{a+b}$	f	$x = a, y = b - a^2$

Exercise A.9.2

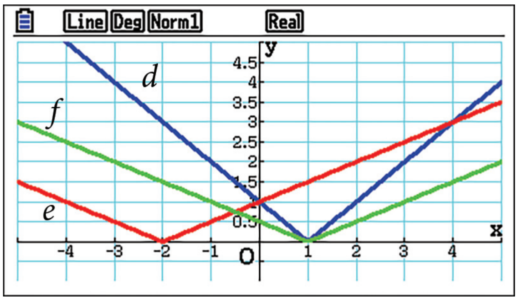
1	a	$x = 4, y = -5, z = 1$	b	$x = 0, y = 4, z = -2$
	c	$x = 10, y = -7, z = 2$	d	$x = 1, y = 2, z = -2$
	e	$\varnothing$	f	$x = 2t - 1, y = t, z = t$
	g	$x = 2, y = -1, z = 0$	h	$\varnothing$

Exercise B.8.1

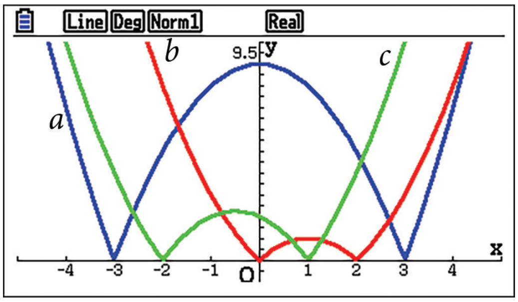
1. a, b, c



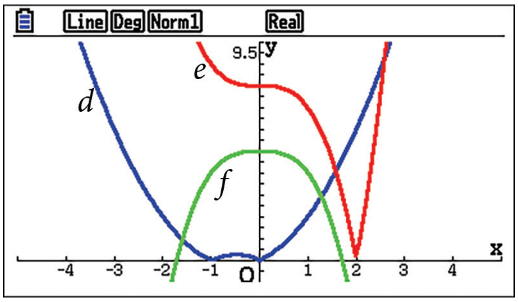
d, e, f



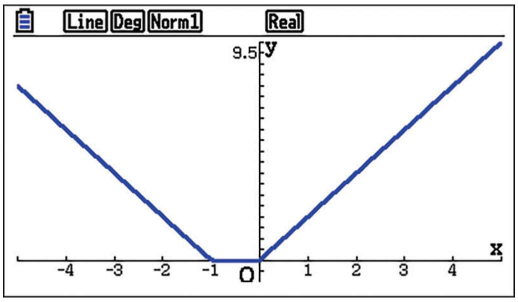
2. a, b, c



d, e, f

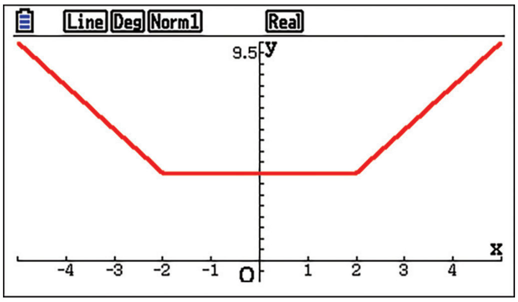


3. a



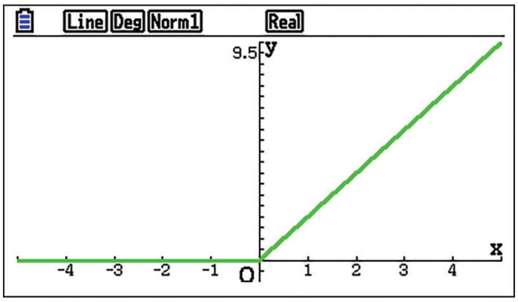
Range: $[0,\infty)$

b



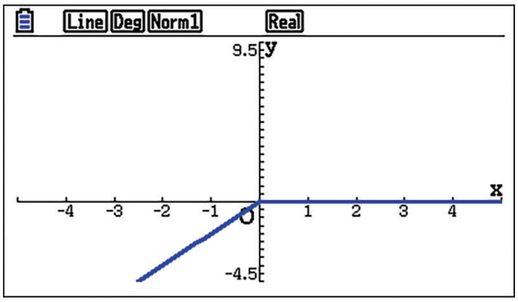
Range: $[4,\infty)$

c



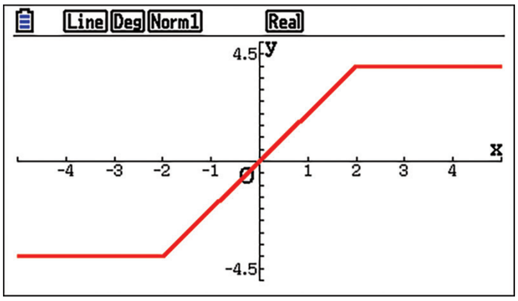
Range: $[0,\infty)$

d



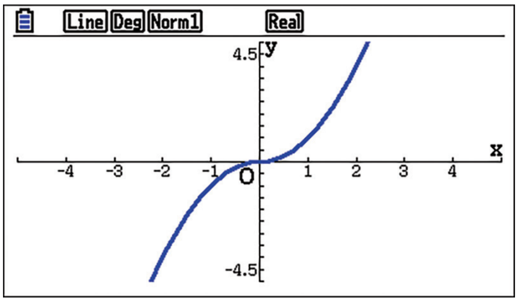
Range: (−∞,0]

e

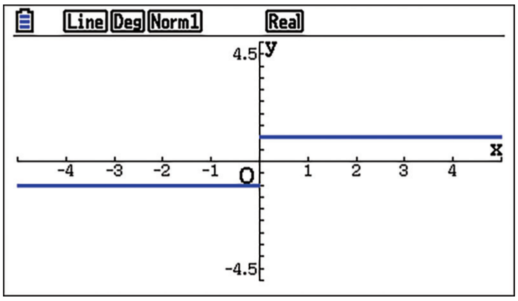


Range: [−4,4]

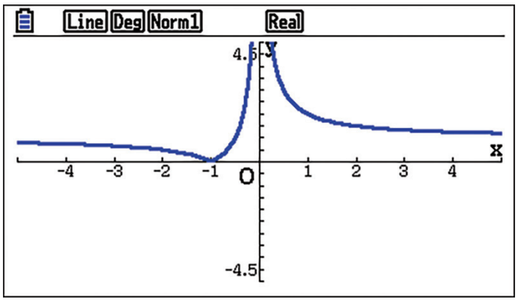
4. a



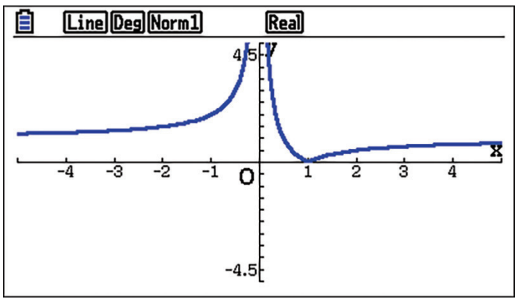
b



c



d



5.
a $x \le 2$
b $x \le -\frac{7}{3}$
c $x > \frac{1}{5}$
d $0 < x < 1$ or $x < -2$
e $-3 < x < 2$
f $-2 < x < 1$
g $x < -1$ or $x > 4$
h $\mathbb{R}$
i always false
j $x < \frac{-1-\sqrt{5}}{2}$ or $x > \frac{-1+\sqrt{5}}{2}$
k $\frac{-1-\sqrt{13}}{2} \le x \le \frac{-1+\sqrt{13}}{2}$
l $1-\sqrt{6} \le x \le 1+\sqrt{6}$

6.
a $-4 \le x \le 4$
b $-3 \le x \le 5$
c $\mathbb{R}$
d $x \le ½$
e $\frac{1-\sqrt{5}}{2} \le x \le \frac{1+\sqrt{5}}{2}$
f $x \ge 0$
g $x \le -2$ or $x \ge 4$
h $x \ge 1$
i $x \le -\frac{3}{2}$ or $x \ge \frac{3}{10}$
j $x \le -2$ or $x \ge \frac{4}{3}$

7.
a $\frac{1-\sqrt{5}}{2} \le x \le \frac{1+\sqrt{5}}{2}$
b $\frac{-1-\sqrt{5}}{2} \le x \le \frac{-1+\sqrt{5}}{2}$
c $1-\sqrt{3} \le x \le 1+\sqrt{3}$ or $x \le 1-\sqrt{5}$ or $x \ge 1+\sqrt{5}$
d $\varnothing$
e $x \le 1$ or $x \ge 5$
f $x \le 2$ or $x \ge 4$
g $\frac{5}{9} \le x \le 1$
h $-3 \le x \le 3$

Toolbox

1.
$$a^3+b^3 \ge ab(a+b)$$
$$a^3+b^3-ab(a+b) \ge 0$$
$$(a+b)(a^2-ab+b^2)-ab(a+b) \ge 0$$
$$(a+b)(a^2-2ab+b^2) \ge 0$$
$$(a+b)(a-b)^2 \ge 0$$

This product is non-negative since each factor is non-negative: $0 < a + b$ since $0 < a, b$ and $(a-b)^2 \ge 0$ for all $a, b \in \mathbb{R}^+$.

Equation holds when $a = b$. $a^3+b^3 \ge ab(a+b) \Leftrightarrow a = b$.

2. $a^2+b^2+c^2 \ge ab+bc+ac$. Multiply the inequality by 2.

$$a^2+b^2-2ab+a^2+c^2-2ac+b^2+c^2-2bc \ge 0$$
$$(a-b)^2+(a-c)^2+(b-c)^2 \ge 0$$

This is true since $(a-b)^2 \ge 0, (a-c)^2 \ge 0, (b-c)^2 \ge 0$.

Their sum is also non-negative.

Equation holds if $a = b = c$.

Alternative solution

Using the inequality of arithmetic and geometric means (AM–GM inequality): $a^2+b^2+c^2 \geq ab+bc+ac$

$\frac{a^2+b^2}{2} \geq ab$ AM–GM inequality for $0 \leq a^2$ and $0 \leq b^2$ and $\sqrt{a^2b^2} = |ab| = \begin{cases} ab \text{ if } a,b \geq 0 \text{ or } a,b \leq 0 \\ -ab \text{ if } a \text{ or } b \text{ is negative} \end{cases}$

$\frac{a^2+c^2}{2} \geq ac$ AM–GM inequality for $0 \leq a^2$ and $0 \leq c^2$ etc…

$\frac{b^2+c^2}{2} \geq bc$ AM–GM inequality for $0 \leq c^2$ and $0 \leq b^2$ etc…

Therefore: $a^2+b^2+c^2 \geq ab+bc+ac$

3.
$$a^2+b^2+c^2+3 \geq 2(a+b+c)$$
$$a^2-2a+1+b^2-2b+1+c^2-2c+1 \geq 0$$
$$(a-1)^2+(b-1)^2+(c-1)^2 \geq 0$$

It holds since non-negative values are added. Equation holds if $a = b = c = 1$.

4. $(a+b)(a+c)(b+c) \geq 8abc$

Setting up inequallites of arithmetic and geometric mean where $a, b, c \geq 0$

$a+b \geq 2\sqrt{ab}$

$a+c \geq 2\sqrt{ac}$

$b+c \geq 2\sqrt{bc}$

The product of the three terms $(a+b)(a+c)(b+c) \geq 2\sqrt{ab} \times 2\sqrt{ac} \times 2\sqrt{bc} \geq 8abc$

$(a+b)(a+c)(b+c) \geq 8abc$

5. $\frac{2ab}{a+b} \geq \frac{a+b}{2}$ Multiplying by $2(a + b)$ will not reverse the inequality sign since $a,b \in \mathbb{R}^+$.

$4ab \leq (a+b)^2$

$0 \leq a^2+b^2-2ab$

$0 \leq (a-b)^2$

It is true for $a,b \in \mathbb{R}^+$ .Equation holds for $a = b$.

6. $\frac{a}{b^2}+\frac{b}{a^2} \geq \frac{1}{a}+\frac{1}{b}$

Multiplying by $0 < a^2$, $0 < b^2$ will not reverse the sign of the inequality.

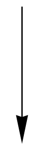
$$a^3+b^3 \geq a^2b+ab^2$$
$$a^3-ab^2+b^3-ba^2 \geq 0$$
$$a(a^2-b^2)+b(b^2-a^2) \geq 0$$
$$(a-b)(a^2-b^2) \geq 0$$
$$(a-b)^2(a+b) \geq 0$$

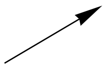
Exercise C.11.1

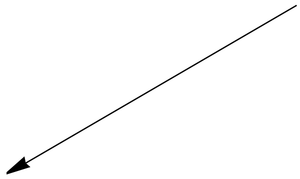
1.	v	2.	s	3.	s	4.	v	5.	v
6.	v	7.	s	8.	s				

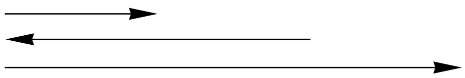
Exercise C.11.2

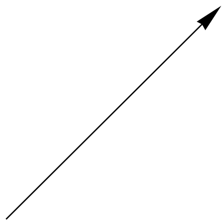
1 **a**

b 

c 

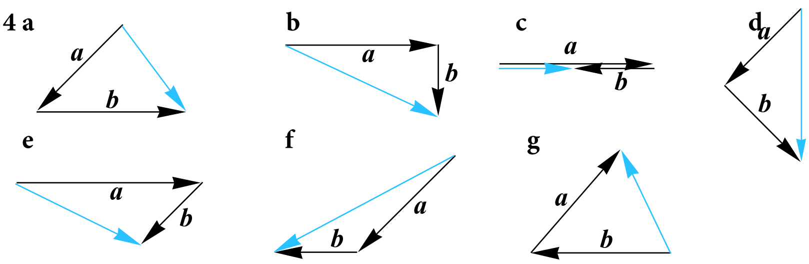
d 

2 **a** **b** **c** 

d 

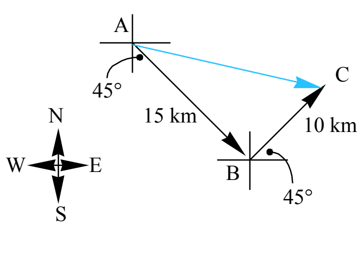
3 **a** {a,b,e,g,u}; {d,f} **b** {d,f}; {a,c}; {b,e} **c** {a,g},{c,g}

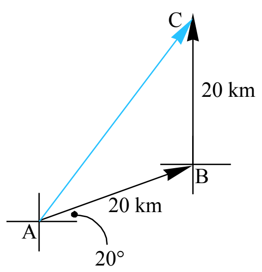
d {d,f}, {b,e} **e** {d,f}, {b,e}, {a,c,g}

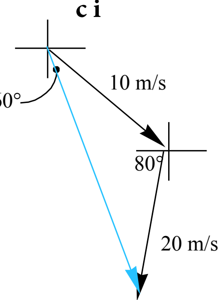
4 

5 **a** AC **b** AB **c** AD **d** BA **e** **0**

6 **a** Y **b** N **c** Y **d** Y **e** N

7 a i 

b i 

c i 

ii $\sqrt{325}$ **iv** $20\sqrt{2(1-\cos 110°)}$ **v** $10\sqrt{5-4\cos 110°}$

8	72.11 N, E $33°41'$ N				
9	2719 N along river				
10	**b i**	200 kph N	**ii**	213.6 kph, N $7°37'$ W	
11	**b i**	200	**ii**	369.32	

Exercise C.11.3

1	**a**	$4\boldsymbol{i}+28\boldsymbol{j}-4\boldsymbol{k}$	**b**	$12\boldsymbol{i}+21\boldsymbol{j}+15\boldsymbol{k}$	**c**	$-2\boldsymbol{i}+7\boldsymbol{j}-7\boldsymbol{k}$	**d**	$-6\boldsymbol{i}-12\boldsymbol{k}$
2	**a**	$3\boldsymbol{i}-4\boldsymbol{j}+2\boldsymbol{k}$	**b**	$-8\boldsymbol{i}+24\boldsymbol{j}+13\boldsymbol{k}$	**c**	$18\boldsymbol{i}-32\boldsymbol{j}+\boldsymbol{k}$	**d**	$-15\boldsymbol{i}+36\boldsymbol{j}+12\boldsymbol{k}$
3	**a**	$\begin{pmatrix}11\\0\\8\end{pmatrix}$	**b**	$\begin{pmatrix}-27\\1\\-22\end{pmatrix}$	**c**	$\begin{pmatrix}-3\\-6\\12\end{pmatrix}$	**d**	$\begin{pmatrix}16\\-1\\14\end{pmatrix}$
4	$\begin{pmatrix}-5\\3\end{pmatrix}$							
5	$\begin{pmatrix}-2\\3\end{pmatrix}$, (−2, 3)							
6	**a**	$8\boldsymbol{i}-4\boldsymbol{j}-28\mathbf{k}$	**b**	$-19\boldsymbol{i}-7\boldsymbol{j}-16\mathbf{k}$	**c**	$-17\boldsymbol{i}+\boldsymbol{j}+22\mathbf{k}$	**d**	$40\boldsymbol{i}+4\boldsymbol{j}-20\mathbf{k}$
7	**a**	$\begin{pmatrix}20\\1\\25\end{pmatrix}$	**b**	$\begin{pmatrix}12\\2\\16\end{pmatrix}$	**c**	$\begin{pmatrix}-4\\-38\\-32\end{pmatrix}$	**d**	$\begin{pmatrix}-20\\-22\\-40\end{pmatrix}$
8	$A=-4, B=-7$							
9	**a**	(2, −5)	**b**	(−4, 3)	**c**	(−6, −5)		

10 Depends on basis used. Here we used: East as $\boldsymbol{i}$, North $\boldsymbol{j}$ and vertically up $\boldsymbol{k}$

b $D = 600\boldsymbol{i}-800\boldsymbol{j}+60\boldsymbol{k}, A = -1200\mathbf{i}-300\boldsymbol{j}+60\boldsymbol{k}$ **c** $1800\boldsymbol{i}-500\boldsymbol{j}$

Exercise C.11.5

1	**a**	4	**b**	−11.49	**c**	25		
2	**a**	12	**b**	27	**c**	−8	**d**	−49
	f	4	**g**	−21	**h**	6	**i**	−4
3	**a**	79°	**b**	108°	**c**	55°	**d**	50°
	e	74°	**f**	172°	**g**	80°	**h**	58°
4	**a**	−8	**b**	0.5				
5	**a**	−6	**b**	2	**c**	Not possible	**d**	5
	e	Not possible	**f**	0				

6 **a** $4-2\sqrt{3}$ **b** $2\sqrt{3}-4$ **c** $14-2\sqrt{3}$ **d** Not possible

7 1

8 105.2°

9 $x = -\frac{16}{7}, y = -\frac{44}{7}$

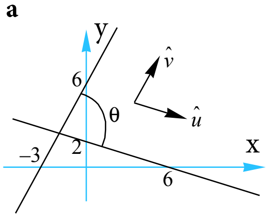
10 $\pm\frac{1}{\sqrt{11}}(-\boldsymbol{i}+\mathbf{j}+3\boldsymbol{k})$

12 **a** $\lambda(-16\boldsymbol{i}-10\boldsymbol{j}+\boldsymbol{k})$ **b** e.g. $\boldsymbol{i}+\boldsymbol{j}+\frac{3}{7}\boldsymbol{k}$

14 $\boldsymbol{a}\perp\boldsymbol{b}-\boldsymbol{c}$ if $\boldsymbol{b}\neq\boldsymbol{c}$ or $\boldsymbol{b} = \boldsymbol{c}$

16 **a** $\frac{1}{3}$ **b** $\frac{1}{\sqrt{3}}$

17 **a**



b i $\hat{\boldsymbol{u}} = \frac{1}{\sqrt{10}}(3\boldsymbol{i}-\mathbf{j})$ **ii** $\hat{\boldsymbol{v}} = \frac{1}{\sqrt{5}}(\boldsymbol{i}+2\mathbf{j})$

c 81.87°

18 $\frac{1}{2}(-\boldsymbol{i}+\mathbf{j}+\sqrt{2}\boldsymbol{k})$

23 **a** Use $\boldsymbol{i}$ as a 1 km eastward vector and $\boldsymbol{j}$ as a 1 km northward vector.

b $\overrightarrow{WD} = 4\boldsymbol{i}+8\boldsymbol{j}$, $\overrightarrow{WS} = 13\boldsymbol{i}+\boldsymbol{j}$ and $\overrightarrow{DS} = 9\boldsymbol{i}-7\boldsymbol{j}$ **c** $\frac{1}{\sqrt{80}}(4\boldsymbol{i}+8\boldsymbol{j})$

d $\frac{d}{\sqrt{80}}(4\boldsymbol{i}+8\boldsymbol{j})$ **e** $3\boldsymbol{i}+6\boldsymbol{j}$

Exercise C.11.6

1 **a i** $r = \boldsymbol{i}+2\boldsymbol{j}$ **ii** $r = -5\boldsymbol{i}+11\boldsymbol{j}$ **iii** $r = 5\boldsymbol{i}-4\boldsymbol{j}$ **b** line joins (1, 2) and (5, –4)

2 **a** $r = 2\boldsymbol{i}+5\boldsymbol{j}+\lambda(3\boldsymbol{i}-4\boldsymbol{j})$ **b** $r = -3\boldsymbol{i}+4\boldsymbol{j}+\lambda(-\boldsymbol{i}+5\boldsymbol{j})$ **c** $r = \boldsymbol{j}+\lambda(7\boldsymbol{i}+8\boldsymbol{j})$

d $r = \boldsymbol{i}-6\boldsymbol{j}+\lambda(2\boldsymbol{i}+3\boldsymbol{j})$ **e** $r = \begin{pmatrix}-1\\-1\end{pmatrix}+\lambda\begin{pmatrix}-2\\10\end{pmatrix}$ or $r = -\boldsymbol{i}-\boldsymbol{j}+\lambda(-2\boldsymbol{i}+10\boldsymbol{j})$

f $r = \begin{pmatrix}1\\2\end{pmatrix}+\lambda\begin{pmatrix}5\\1\end{pmatrix}$ or $r = \boldsymbol{i}+2\boldsymbol{j}+\lambda(5\boldsymbol{i}+\boldsymbol{j})$

3 **a** $r = 2\boldsymbol{i}+3\boldsymbol{j}+\lambda(2\boldsymbol{i}+5\boldsymbol{j})$ **b** $r = \boldsymbol{i}+5\boldsymbol{j}+\lambda(-3\boldsymbol{i}-4\boldsymbol{j})$ **c** $r = 4\boldsymbol{i}-3\boldsymbol{j}+\lambda(-5\boldsymbol{i}+\boldsymbol{j})$

4 **a** $r = 9\boldsymbol{i}+5\boldsymbol{j}+\lambda(\boldsymbol{i}-3\boldsymbol{j})$ **b** $r = 6\boldsymbol{i}-6\boldsymbol{j}+t(-4\boldsymbol{i}-2\boldsymbol{j})$

c $r = -\boldsymbol{i}+3\boldsymbol{j}+\lambda(-4\boldsymbol{i}+8\boldsymbol{j})$ **d** $r = \boldsymbol{i}+2\boldsymbol{j}+\mu\left(\frac{1}{2}\boldsymbol{i}-\frac{1}{3}\boldsymbol{j}\right)$

5 **a** $x = -8+2\mu$, $y = 10+\mu$ **b** $x = 7-3\mu$, $y = 4-2\mu$

c $x = 5 + 2.5\mu$, $y = 3 + 0.5\mu$ **d** $x = 0.5 - 0.1t$, $y = 0.4 + 0.2t$

6 **a** $\frac{x-1}{3} = y - 3$ **b** $\frac{x-2}{-7} = \frac{y-4}{-5}$

c $x + 2 = \frac{y+4}{8}$ **d** $x - 0.5 = \frac{y-0.2}{-11}$

e $x = 7$

7 **a** $r = 2\boldsymbol{j} + t(3\boldsymbol{i} + \mathbf{j})$ **b** $r = 5\boldsymbol{i} + t(\boldsymbol{i} + \boldsymbol{j})$ **c** $r = -6\boldsymbol{i} + t(2\boldsymbol{i} + \boldsymbol{j})$

8 **a** $6\boldsymbol{i} + 13\boldsymbol{j}$ **b** $-\frac{16}{3}\boldsymbol{i} - \frac{28}{3}\boldsymbol{j}$

9 $r = 2\boldsymbol{i} + 7\boldsymbol{j} + t(4\boldsymbol{i} + 3\boldsymbol{j})$

11 **a** (4, –2), (–1, 1), (9, –5) **b** –2 **d** $r = 4\boldsymbol{i} - 2\boldsymbol{j} + \lambda(-5\boldsymbol{i} + 3\boldsymbol{j})$ **e i** $M \parallel L$ **ii** $M = L$

12 $4x + 3y = 11$

13 **a** $\frac{-3}{\sqrt{13}}, \frac{2}{\sqrt{13}}$ **b** $\frac{4}{5}, \frac{3}{5}$

14 **b** ii and iii

15 (–83, –215)

16 $r = \frac{k}{7}(19\boldsymbol{i} + 20\boldsymbol{j})$

17 **a** $\left(\frac{92}{11}, \frac{31}{11}\right)$ **b** Ø **c** Lines are coincident, all points are common.

Exercise C.11.7

1 **a** $r = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k} + t(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$ **b** $r = 2\boldsymbol{i} - 3\boldsymbol{j} - \boldsymbol{k} + t(-2\boldsymbol{i} + \boldsymbol{k})$

2 **a** $r = 2\boldsymbol{i} + 5\boldsymbol{k} + t(\boldsymbol{i} + 4\boldsymbol{j} + 3\boldsymbol{k})$ **b** $r = 3\boldsymbol{i} - 4\boldsymbol{j} + 7\boldsymbol{k} + t(4\boldsymbol{i} + 9\boldsymbol{j} - 5\boldsymbol{k})$

c $r = 4\boldsymbol{i} + 4\boldsymbol{j} + 4\boldsymbol{k} + t(7\boldsymbol{i} + 7\boldsymbol{k})$

3 **a** $\frac{x}{3} = \frac{y-2}{4} = \frac{z-3}{5}$ **b** $\frac{x+2}{5} = \frac{z+1}{-2}$, $y = 3$ **c** $x = y = z$

4 $\begin{aligned} x &= 5-7t \\ y &= 2+2t \\ z &= 6-4t \end{aligned}$ $\quad \boldsymbol{r} = \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix} + t\begin{pmatrix} -7 \\ 2 \\ -4 \end{pmatrix}$ $\quad \frac{x-5}{-7} = \frac{y-2}{2} = \frac{z-6}{-4}$

5 $\left(\frac{13}{5}, \frac{23}{5}, 0\right)$

6 **a** $\begin{aligned} x &= 2+3t \\ y &= 5+t \\ z &= 4+0.5t \end{aligned}$ **b** $\begin{aligned} x &= 1+1.5t \\ y &= t \\ z &= 4-2t \end{aligned}$

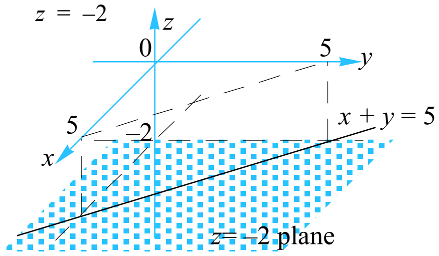
c $\begin{aligned} x &= 3-t \\ y &= 2-3t \\ z &= 4+2t \end{aligned}$ **d** $\begin{aligned} x &= 1+2t \\ y &= 3+2t \\ z &= 2+0.5t \end{aligned}$

7 **a** $\frac{x-4}{3} = \frac{y-1}{-4} = \frac{z+2}{-2}$ **b** $x = 2, y = \frac{z-1}{-3}$

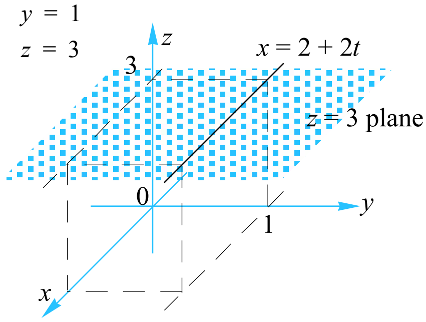
9 **a** $\frac{x+1}{2} = y-3 = \frac{z-5}{-1}$ **b** $\frac{x-2}{2} = \frac{z-1}{-2}, y = 1$

10 **a** $(1, -1, 0)$ **b** $a = 15, b = -11$

11 a $\begin{aligned} x &= 1+t \\ y &= 4-t \\ z &= -2 \end{aligned}$



b $\begin{aligned} x &= 2+2t \\ y &= 1 \\ z &= 3 \end{aligned}$



12 $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0.5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1.5 \\ 1 \end{pmatrix}$. Line passes through (1, 0.5, 2) and is parallel to the vector $2\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + \boldsymbol{k}$

13 **a** 54.74° **b** 82.25° **c** 57.69°

14 **a** (4, 10.5, 15) **b** Does not intersect.

15 **a** $L: x = \frac{y-2}{2} = \frac{z}{5}, M: \frac{x+1}{2} = \frac{y+1}{3} = \frac{z-1}{-2}$ **b** Ø **c** 84.92°

d i $(0, 2, 0)$ **ii** $\left(0, \frac{1}{2}, 0\right)$

18 $\frac{x}{4} = \frac{y}{9} = \frac{z}{3}$

19 $k = -\frac{7}{2}$

20 64°

21 3 or –2

22 $12\boldsymbol{i} + 6\boldsymbol{j} - 7\boldsymbol{k}$ (or any multiple thereof)

23 Not parallel. Do not intersect. Lines are skew.

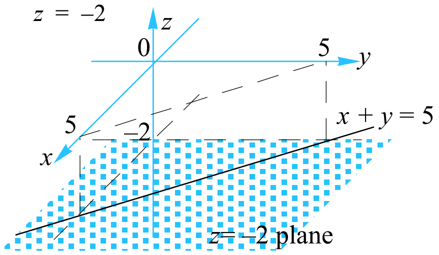
Exercise C.11.8

1 **a** $\frac{x+1}{2} = y - 3 = \frac{z-5}{-1}$ **b** $\frac{x-2}{2} = \frac{z-1}{-2}, y = 1$

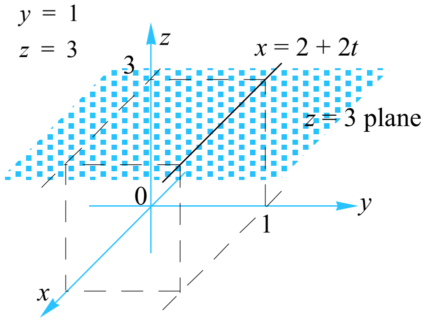
2 **a** $(1, -1, 0)$ **b** $a = 15, b = -11$

3

a $x = 1 + t$
$y = 4 - t$
$z = -2$



b $x = 2 + 2t$
$y = 1$
$z = 3$



4 $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0.5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1.5 \\ 1 \end{pmatrix}$. Line passes through (1, 0.5, 2) and is parallel to the vector $2\boldsymbol{i} - \frac{3}{2}\boldsymbol{j} + \boldsymbol{k}$

5 **a** 54.74° **b** 82.25° **c** 57.69°

6 **a** (4, 10.5, 15) **b** Does not intersect.

7 **a** $L: x = \frac{y-2}{2} = \frac{z}{5}, M: \frac{x+1}{2} = \frac{y+1}{3} = \frac{z-1}{-2}$ **b** Ø **c** 84.92°

d i $(0, 2, 0)$ **ii** $\left(0, \frac{1}{2}, 0\right)$

10 $\frac{x}{4} = \frac{y}{9} = \frac{z}{3}$

11 $k = -\frac{7}{2}$

12 64°

13 3 or –2

14 $12\boldsymbol{i} + 6\boldsymbol{j} - 7\boldsymbol{k}$ (or any multiple thereof)

15 Not parallel. Do not intersect. Lines are skew.

16 $t=2$

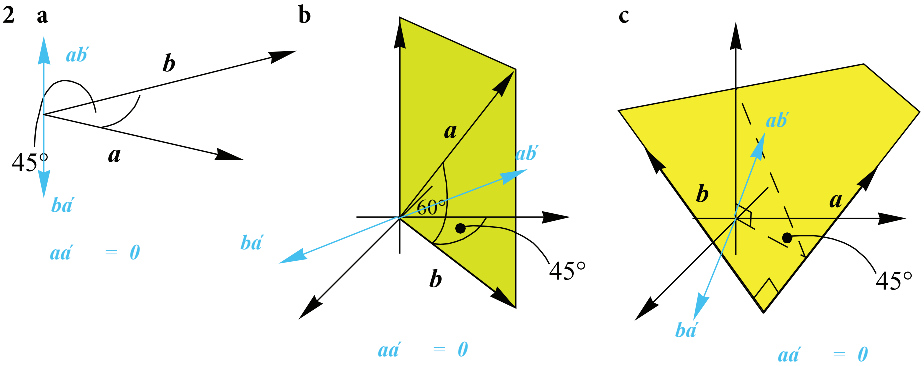
17 (1,1,1), no

18 ~21

Exercise C.11.9

1 **a** 5 **b** $4\sqrt{3}$ **c** 0 **d** 6 **e** 0

2 **a** **b** **c**



3 **a** $2\sqrt{91}$ **b** 16

4 56°27′

5 $3\sqrt{5}$

6 **a** $2\sqrt{3}$ **b** $5\sqrt{3}$

Exercise C.11.10

1 **a** $-12\boldsymbol{i}+4\boldsymbol{k}$ **b** $10\boldsymbol{i}-2\boldsymbol{j}-2\boldsymbol{k}$ **c** $18\boldsymbol{i}-9\boldsymbol{j}$ **d** $10\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}$

e $-6\boldsymbol{i}+9\boldsymbol{j}+8\boldsymbol{k}$ **f** $20\boldsymbol{i}-13\boldsymbol{j}-4\boldsymbol{k}$

2 $-10\boldsymbol{i}+6\boldsymbol{j}-2\boldsymbol{k}$

5 **a i** 0 **ii** 0

6 $\dfrac{-6\boldsymbol{i}+2\boldsymbol{j}-2\boldsymbol{k}}{\sqrt{11}}$

7 **a** $\lambda\boldsymbol{k}$ **b** $\lambda(9\boldsymbol{i}-3\boldsymbol{j}+9\boldsymbol{k})$

8 **a** 90° **b** 79.1°

12 They must be parallel.

Exercise C.11.11

1 **a** $\sqrt{54}$ **b** $\sqrt{234}$

2 **a** $\frac{1}{2}(-3\boldsymbol{i}-13\boldsymbol{j}+29\boldsymbol{k})$ **b** $\frac{1}{2}\sqrt{1019}$ **c** 67.84°

3 $\frac{1}{2}\sqrt{2331}$

4 $12\sqrt{2}$

5 $43°36'$

6 $\sqrt{293}$ sq. units

7 $\frac{1}{2}\sqrt{35}$

9 **a** $\mathbf{OA} = \cos\alpha\boldsymbol{i}+\sin\alpha\boldsymbol{j}, \mathbf{OB} = \cos\beta\boldsymbol{i}+\sin\beta\boldsymbol{j}$

12 66 cubic units

13 **b** $k = 0.5$

Exercise C.11.12

1 **a** $r = \boldsymbol{i}+\boldsymbol{k}+\lambda(3\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k})+\mu(-2\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})$ **b** $r = -\boldsymbol{i}+2\boldsymbol{j}+\boldsymbol{k}+\lambda(\boldsymbol{i}-\boldsymbol{j}+2\boldsymbol{k})+\mu(-\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})$

c $r = 4\boldsymbol{i}+\boldsymbol{j}+5\boldsymbol{k}+\lambda(2\boldsymbol{i}+2\boldsymbol{j}-\boldsymbol{k})+\mu(2\boldsymbol{i}-\boldsymbol{j}+3\boldsymbol{k})$ **d** $r = 2\boldsymbol{i}-3\boldsymbol{j}-\boldsymbol{k}+\lambda(-3\boldsymbol{i}+\boldsymbol{j}-2\boldsymbol{k})+\mu\left(\boldsymbol{i}-2\boldsymbol{j}+\frac{1}{2}\boldsymbol{k}\right)$

2 **a** $3x-5y+z = 4$ **b** $x-3y-2z = -9$ **c** $5x-8y-6z = -18$

d $7x+y-10z = 21$

3 **a i** $\mathbf{r} = \begin{pmatrix}2\\3\\4\end{pmatrix}+\lambda\begin{pmatrix}-3\\-1\\-3\end{pmatrix}+\mu\begin{pmatrix}-2\\2\\2\end{pmatrix}$ **ii** $x+3y-2z = 3$

b i $\mathbf{r} = \begin{pmatrix}3\\-1\\5\end{pmatrix}+\lambda\begin{pmatrix}-2\\5\\-11\end{pmatrix}+\mu\begin{pmatrix}-1\\4\\-1\end{pmatrix}$ **ii** $13x+3y-z = 31$

4 **a i** $\mathbf{r} = \begin{pmatrix}2\\-2\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\-1\end{pmatrix}+\mu\begin{pmatrix}3\\1\\2\end{pmatrix}$ **ii** $\mathbf{r} = \lambda\begin{pmatrix}2\\-1\\-1\end{pmatrix}+\mu\begin{pmatrix}3\\1\\2\end{pmatrix}$

b i $x+7y-5z = -27$ **ii** $x+7y-5z = 0$ **c** $-\mathbf{i}-7\mathbf{j}+5\mathbf{k}$

d Coefficients are the negative of those in part **b.**

Exercise C.11.13

1 **a** $2x - y + 5z = 7$ **b** $-4x + 6y - 8z = 34$ **c** $-x + 3y - 2z = 0$

d $5x + 2y + z = 0$

2 c and d

3 **a** $-3x - y + 2z = 3$ **b** $y = 2$ **c** $2x + 2y - z = -3$

4 **a** 29.5° **b** 70° **c** 90° **d** 11°

5 **a** 83° **b** 50° **c** 49°

6 **a** $2x + y + 2z = 12$ **b** $8x + 17y - z = 65$

7 $x - 2y + 3z = -2$

8 $3x - 2y + 5z = -2$

9 $a = \frac{24}{13}, b = \frac{18}{13}$

10 **a** $r = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k} + t(2\boldsymbol{i} + 5\boldsymbol{j} + 5\boldsymbol{k})$ **b** $3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ **c** 49.8°

Exercise C.11.14

1 **a** $2x + 7y - 6z = 33 ; \boldsymbol{r} \bullet \begin{pmatrix} 2 \\ 7 \\ -6 \end{pmatrix} = 33$ **b** $x - 2y = 0 ; \boldsymbol{r} \bullet \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = 0$

2 $3x - y - z = 2$

5 **a** 3 **b** $\frac{5}{3}$ **c** $\sqrt{11}$ **d** $\frac{20}{\sqrt{21}}$

6 $3x + 4y - 5z = -4$

7 $2x + 3y - 3z = 5$

9 $x + 5y - 6z = -19$

Exercise C.11.15

1 **a** (7, 5, –3)

2 Lines that intersect are **b** and **c**; (7, –4, 10); 46.7°

3 (5, –2, –3)

4 (4, 0, 6)

Exercise C.11.16

1 **a i** (7, 4, 2) **ii** 36.3° **b i** (5, 2, –5) **ii** 10.1°

c i (6, –5, –7) **ii** 4.4° **d i** (3, –1, 1) **ii** 29.1°

2 **a** $\left(\frac{3}{2}, \frac{5}{2}, 2\right)$ **b** (0, 4, 1)

3 **a** Plane is parallel to the z-axis slicing the x-y plane on the line $x + y = 6$.

b $x = 4$ forms a plane. $y = 2z$ is in this plane parallel to the y-z plane. (4, 2, 1)

4 13

Exercise C.11.17

1 e.g. the faces of a triangular prism.

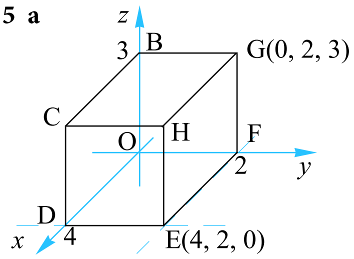
2 **a** $4 - 2x = \frac{2y - 4}{-5} = z$ or $\frac{x}{1} = \frac{y + 8}{5} = \frac{z - 4}{-2}$

3 **a** $5 - 4x = y = \frac{8 - 4z}{7}$ or $\frac{x}{1} = \frac{y - 5}{-4} = \frac{z + 6}{7}$

4 **a** No solution **b** Unique solution (5, 1, 4) **c** Unique solution (5, 1, −3)

d Intersect on plane $\frac{5x + 19}{-8} = \frac{5y - 13}{1} = z$

5 **a**



b $x = 4t$ $x = 4s$; $y = 2t$ $y = 2s$; $z = 3t$ $z = 3 - 3s$ (2, 1, 1.5)

c $3x + 6y - 4z = 12$

d $\left(\frac{8}{3}, \frac{4}{3}, 1\right)$ $58°52'$ **e** 59.2°

6 None of these planes is parallel but the lines of intersection of pairs of planes are skew.

7 $k = 2$; $\boldsymbol{r} = \begin{pmatrix} 0 \\ 3.5 \\ 1.5 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2.5 \\ -0.5 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} 3 \\ -4 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} -2 \\ 5 \\ 1 \end{pmatrix}$

8 **a** $-2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ **b** $\boldsymbol{r} = t\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$ **c i** 5 **iii** not 5

9 **b** $(a - b + c, a + b - c, -a + b + c)$ **c** $\left(\frac{1}{a} - \frac{1}{b} + \frac{1}{c}, \frac{1}{a} + \frac{1}{b} - \frac{1}{c}, -\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

10 **a** 2, 3 **b** 3 **c** For $k = 2$, $x - 1 = 4 - y = z$

Exercise D.7.1

1	**a**	0.042	**b**	0.7143			
2	**a**	0.4667	**b**	0.3868			
3	**a**	$\frac{5}{7}$	**b**	$\frac{9}{13}$			
4	$\frac{5}{9}$						
5	**bi**	$\frac{1}{40}$	**ii**	0.2			
6	**ai**	$\frac{2N-\mathrm{m}}{2N}$	**ii**	$\frac{2(N-m)}{2N-m}$	**b**	$\frac{m}{m+(N-m)2^n}$	
7	$\frac{9}{19}$						
8	**a**	0.07	**b**	0.3429	**c**	0.30	**d** 0.0282
9	**a**	0.8008	**b**	0.9767	**c**	0.0003	
10	**a**	0.0464	**b**	0.5819	**c**	0.9969	
11	$\frac{1}{31}$						
12	$\frac{10}{31}$						
13	**a**	0.8	**b**	0.005			
14	$\frac{10}{21}$						
15	M_1						
16	**a**	0.57	**b**	$\frac{18}{57}$			

Exercise D.8.1

1. a $\frac{12}{25}$ b $\frac{6}{25}$ c $\frac{22}{25}$

2. $^1/_6$

3. a

x	2	3	4	5	6	7	8	9	10	11	12
P(X=x)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Note that may of these fractions will cancel.

b. $^1/_{12}$

4. ~1.85×10^{-14}

5. a $^3/_{14}$ b $^3/_{28}$

6. a 4,5,6,7,8. b P(4)=¼, P(5)=$^3/_{10}$, P(6)=$^{29}/_{100}$, P(7)=$^3/_{25}$, P(8) =$^1/_{25}$.

7. $^{144}/_{205}$

8. 0.143

11. a $P(X=x)=\left(\frac{17}{18}\right)^{x-1}\left(\frac{1}{18}\right), x=1,2,3...$ b 0.842

Exercise D.8.2

1. k = $^1/_9$, $^1/_{27}$

2. 0.1

3. k = 8, $^5/_{16}$

4. 0.05637

5. b 0.0067

6. $\frac{1}{\ln 3}$ b 0.369

7. b 0.135 c 1.8% d 3

8. a 0.75 b 0.269 c 0.1495 d 0.575

Exercise D.8.3

1. a Both 0.5 b Variance = $^1/_{12}$; SD ≈ 0.2887

2. a Mode = 1; Mean = 0.75; Median ≈ 0.7937 b Var. = 0.0375; SD ≈ 0.1936

3. a All 5 g b Var. = 0.2; SD ≈ 0.447 c [4.1065, 5.894]

4. a Mode = 2, Mean = , Median ≈ 1.414 b SD ≈ 0.4714 c Mean

5. a 0 b ≈ 3.5 s c 5 s d 5 s e 15 s

6. All π.

7. Mean = 8.5 cm Var. = 1.25.

8. a $80\ln\left(\frac{4}{3}\right) \approx 23.01$ b $F(t) = 1 - e^{-t/80}, t \geq 0$ c Use graphics calculator.

9. b $a = \frac{3}{7}, b = \frac{5}{2}$ c mode = 1.25

10. a $k = \frac{1}{2(\sqrt{5}-1)}$ b $\frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}-1} \approx 0.4078$ c 4.62 days d 4.75 days

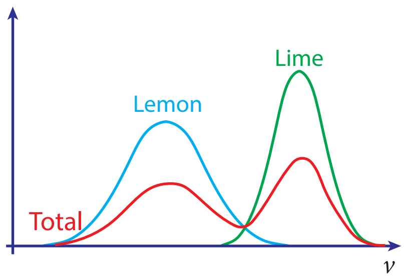
11. a $(1-x)^{3/2} - \frac{3}{2}x\sqrt{1-x}$ b i 1.5 ii $1-\left(\frac{1}{2}\right)^{2/3} \approx 0.37$; 0 iii 0.4

12. a E(X) = 4.2, Var(X) = 3.69 b E(X) = 2.25, Var(X) = 1.34
 c E(X) = ln(2)+$^4/_3$, Var(X) = 0.4767

Exercise D.8.4

1. Mean = 116.76 STD = 12.15 kJ

2.



3. μ = 4.2, σ = 0.6

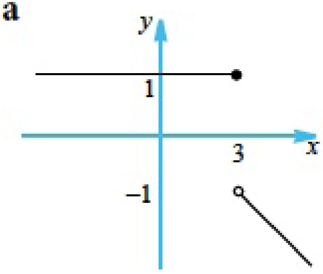
4. μ = 104 kph, σ = 4.8 kph

5. μ = 61.8 g, σ = 1.58 g

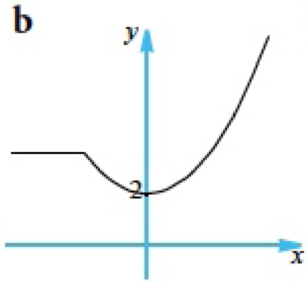
6. 68 kph ≈ 18.89 mps 0.05294 s b 17 m/s c μ = 61.2 kph σ = 4.54 kph

Exercise E.7.1

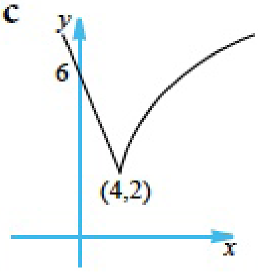
1 a



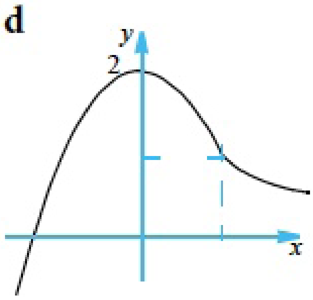
b



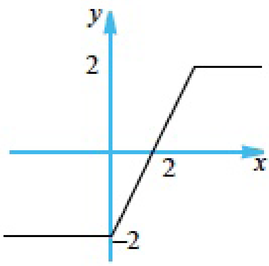
c



d



2 a



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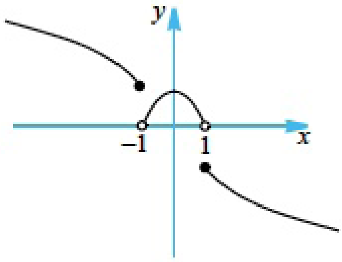

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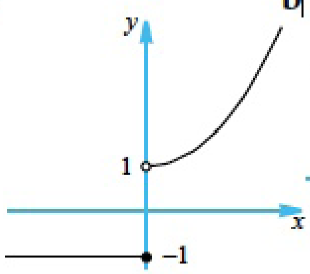

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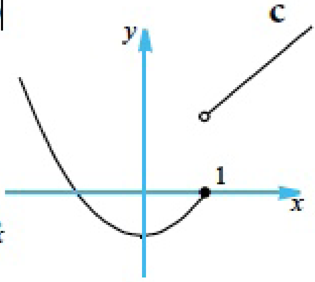

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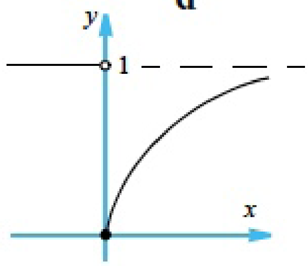

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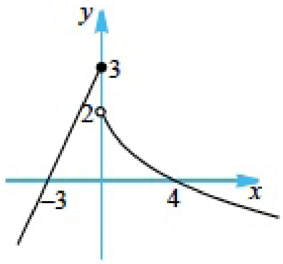

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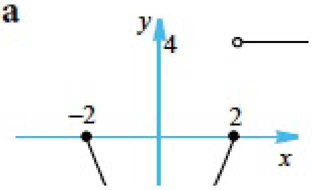

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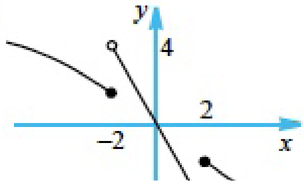
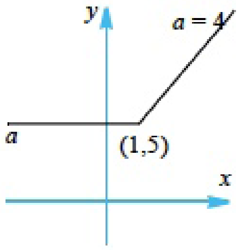


6 a



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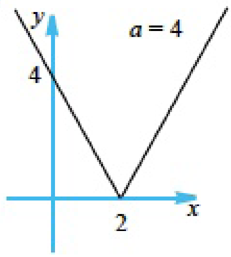
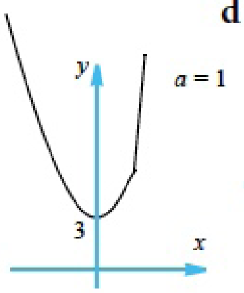


c



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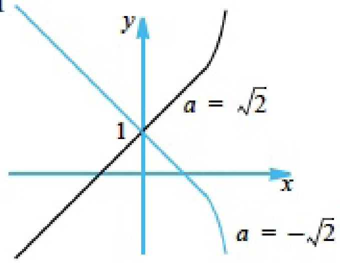
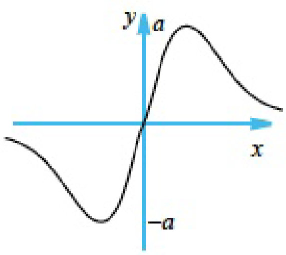
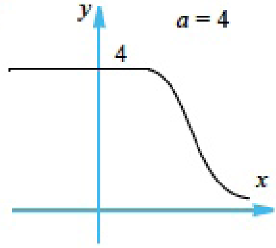


7 a



b



Exercise E.7.2

1. a No b No c Yes d Yes

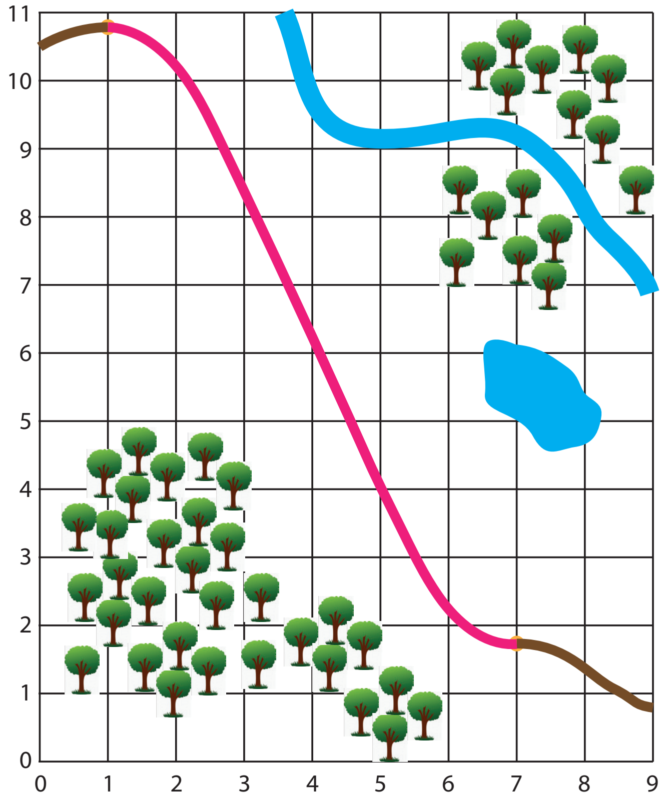
   e No f No

2. $a = 2, b = -3$.

3. $a = 2, n = 2$.

4. $g(x) = e^{2|x|}, x \in \mathbb{R}$ is continuous but not differentiable at $x = 0$.

7. Answers may differ. However: $y = \frac{x^3}{12} - x^2 + \frac{7}{4}x + 15, 1 \le x \le 7$ gives:



Exercise E.7.3

1 **a** $20x^3$ **b** $48(1+2x)^2$ **c** $\frac{2}{x^3}$ **d** $\frac{2}{(1+x)^3}$

e 2 **f** $\frac{6}{(x-2)^3}$ **g** $\frac{42}{x^8}$ **h** $24(1-2x)$

i $-\frac{1}{x^2}$ **j** $\frac{2(x^2+1)}{(1-x^2)^2}$ **k** $-16\sin 4\theta$ **l** $2\cos x - x\sin x$

m $6x^2\cos x + 6x\sin x - x^3\sin x$ **n** $\frac{1}{x}$ **o** $\frac{10}{(2x+3)^3}$

p $6xe^{2x} + 12x^2e^{2x} + 4x^3e^{2x}$ **q** $\frac{8\sin 4x - 15\cos 4x}{e^x}$

r $2\cos x^2 - 4x^2\sin x^2$ **s** $\frac{-48(x^2+2x^5)}{(4x^3-1)^3}$ **t** $\frac{10}{(x-3)^3}$

2 **a** $\frac{-2x}{(x^2+1)^2}$ **b** $\frac{x}{(1-x^2)^{3/2}}$ **c** $\frac{-x}{(1-x^2)^{3/2}}$ **d** $\frac{2}{(x^2+1)^2}$

e $\frac{2x-1}{4(x-x^2)^{3/2}}$ **f** $\frac{2x-3x^2}{4\sqrt{(x^3-x^2)^3}}$

3 $\frac{6\ln x - 5}{x^4}, \frac{n^2\ln x + n\ln x - 2n - 1}{x^{n+2}}$

4 $f'(x) = -\frac{1}{(x+1)^2}, f''(x) = \frac{2}{(x+1)^3}, f^{(iii)}(x) = -\frac{6}{(x+1)^4}, f^{(iv)}(x) = \frac{24}{(x+1)^5}$

$f^{(v)}(x) = -\frac{120}{(x+1)^6}, \ldots\ldots, f^{(n)}(x) = (-1)^n\frac{n!}{(x+1)^{n+1}}$

5 $f(x) = \left(\frac{x+1}{x-1}\right)^n \Rightarrow f''(x) = \frac{4n(n+x)}{(x^2-1)^2}\left(\frac{x+1}{x-1}\right)^n$

7 **a** $a^n e^{ax}$ **b** $\frac{(-1)^n 2^n n!}{(2x+1)^{n+1}}$

c $n = 2k:\quad y^{(n)}(x) = (-1)^k a^{2k}\sin(ax+b),\ k = 1, 2, \ldots$
$n = 2k-1:\quad y^{(n)}(x) = (-1)^{k+1}a^{2k-1}\cos(ax+b),\ k = 1, 2, \ldots$

8 **a** $2 + \frac{1}{8\sqrt{2}}$ **b** $\frac{3+\pi}{2}$

9 -1 [0,1.0768[$\cup$]3.6436,2p]

Exercise E.7.4

1 a $-\frac{15}{16}x^{-7/2} + 240x^{-6}$ **b** $-24t^{-5}$ **c** $1920(2x-1)^{-5}$

d $81\cos(3x) + 16\sin(2x)$ **e** $\frac{15}{16}(x+2)^{-7/2}$ **f** $\frac{24}{(x+1)^5}$

g $81e^{-3x} - \frac{6}{x^4}$ **h** $-119e^{2x}\cos(3x) + 120e^{2x}\sin(3x)$ **l** $\frac{24\ln(x)}{x^5} - \frac{50}{x^5}$

2 **a** $x\sin(x) - 3\cos(x)$ **b** $-4\cos(2x)$

c $-16\sin(4x)$ **d** $48\tan^4(2x) + 64\tan^2(2x) + 16$

3 -3

4 0

5 **a** $(\ln(2))^3 - \ln(2)$ **b** $\frac{2}{\ln(3)}$ **c** $(\ln 3)^2[3 + \ln(3)]$

6 $64(\ln(2))^4$

Exercise E.8.1

1. (a) $\lim_{x\to 0}\left(\frac{x+\sin(2x)}{x-\sin(2x)}\right)=\frac{0}{0}$, so we apply

L'Hospital's rule:

$$\lim_{x\to 0}\left(\frac{x+\sin(2x)}{x-\sin(2x)}\right)=\lim_{x\to 0}\left(\frac{1+2\cos(2x)}{1-2\cos(2x)}\right)$$

$$=\frac{1+2}{1-2}=\underline{\underline{-3.}}$$

(b) $\lim_{x\to \pi}\left(\frac{x-\pi}{\sin(x)}\right)=\frac{0}{0}$, so we apply

L'Hospital's rule:

$$\lim_{x\to \pi}\left(\frac{x-\pi}{\sin(x)}\right)=\lim_{x\to \pi}\left(\frac{1}{\cos(x)}\right)=\frac{1}{1}=\underline{\underline{1.}}$$

(c) $\lim_{x\to \pi/2}\left(\frac{\sin(2x)}{\cos(x)}\right)=\frac{0}{0}$, so we apply

L'Hospital's rule:

$$\lim_{x\to \frac{\pi}{2}}\left(\frac{\sin(2x)}{\cos(x)}\right)=\lim_{x\to \pi/2}\left(\frac{2\cos(2x)}{-\sin(x)}\right)=\frac{-2}{-1}=\underline{\underline{2.}}$$

2. (a) $\lim_{x\to \infty}\left(\frac{x}{e^{2x}}\right)=\frac{\infty}{\infty}\Rightarrow$ L'Hospital's rule.

$$\lim_{x\to \infty}\left(\frac{x}{e^{2x}}\right)=\lim_{x\to \infty}\left(\frac{1}{2e^{2x}}\right)=\frac{1}{\infty}=\underline{\underline{0.}}$$

(b) $\lim_{x \to \infty} \left(\frac{\ln(x)}{x} \right) = \frac{\infty}{\infty} \Rightarrow$ L'Hospital's rule

$$\lim_{x \to \infty} \left(\frac{\ln(x)}{x} \right) = \lim_{x \to \infty} \left(\frac{1/x}{1} \right) = \lim_{x \to \infty} \left(\frac{1}{x} \right) = \underline{\underline{0.}}$$

(c) $\lim_{x \to \infty} \left(\frac{2x}{x + \ln(x)} \right) = \frac{\infty}{\infty} \Rightarrow$ L'Hospital's rule.

$$\lim_{x \to \infty} \left(\frac{2x}{x + \ln(x)} \right) = \lim_{x \to \infty} \left(\frac{2}{1 + 1/x} \right) = \underline{\underline{2.}}$$

3. (a) $\lim_{x \to 0} \left(\frac{2x}{x + \sin(x)} \right) = \frac{0}{0} \Rightarrow$ L'Hospital's rule.

$$\lim_{x \to 0} \left(\frac{2x}{x + \sin(x)} \right) = \lim_{x \to 0} \left(\frac{2}{1 + \cos(x)} \right)$$

$$= \frac{2}{1+1} = \underline{\underline{1.}}$$

(b) $\lim_{x \to 0} \left(\frac{\cos(x) - 1}{x^2} \right) = \frac{0}{0} \Rightarrow$ L'Hospital's rule.

$$\lim_{x \to 0} \left(\frac{\cos(x) - 1}{x^2} \right) = \lim_{x \to 0} \left(\frac{-\sin(x)}{2x} \right) = \frac{0}{0}$$

$\Rightarrow$ L'Hospital's rule again.

$$\lim_{x \to 0} \left(\frac{-\sin(x)}{2x} \right) = \lim_{x \to 0} \left(\frac{-\cos(x)}{2} \right)$$

$$= \underline{\underline{-\frac{1}{2}}}.$$

(c) $\lim_{x \to 0} \left(\frac{x - \sin(x)}{x^3} \right)$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule} \right]$

$= \lim_{x \to 0} \left(\frac{1 - \cos(x)}{3x^2} \right)$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule} \right]$

$= \lim_{x \to 0} \left(\frac{\sin(x)}{6x} \right)$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule} \right]$

$= \lim_{x \to 0} \left(\frac{\cos(x)}{6} \right)$

$= \underline{\underline{\frac{1}{6}}}.$

4. (a) $\lim_{x \to \frac{\pi}{2}} \left(\frac{\sin(x) - 1}{\cos(x)} \right)$ $\left[\frac{0}{0} \Rightarrow \text{L'Hospital's rule} \right]$

$= \lim_{x \to \frac{\pi}{2}} \left(\frac{\cos(x)}{-\sin(x)} \right)$

$= \frac{0}{-1} = \underline{\underline{0.}}$

(b) $\lim_{x \to 0^+} x \ln\left(1 + \frac{1}{x}\right)$

$= \lim_{x \to 0^+} \left(\frac{\ln\left(1 + \frac{1}{x}\right)}{\frac{1}{x}} \right)$ $\left[= \frac{\infty}{\infty} \Rightarrow \text{L'Hospital's rule} \right]$

$= \lim_{x \to 0^+} \left(\frac{\frac{1}{1 + \frac{1}{x}} \left(-\frac{1}{x^2} \right)}{-\frac{1}{x^2}} \right)$

$= \lim_{x \to 0^+} \left(\frac{1}{1 + \frac{1}{x}} \right) = \frac{1}{\infty} = \underline{\underline{0.}}$

(c) $\lim_{x\to 1} \dfrac{l(x)-(x-1)}{x-1}$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 1} \dfrac{\frac{1}{x}-1}{1} = \dfrac{1-1}{1} = \underline{\underline{0}}.$

5(a) $\lim_{x\to \pi/2} \left(\tan(x)+\sec(x)\right) = \lim_{x\to \frac{\pi}{2}} \left(\dfrac{\sin(x)}{\cos(x)} + \dfrac{1}{\cos(x)}\right)$

$= \lim_{x\to \frac{\pi}{2}} \left(\dfrac{\sin(x)+1}{\cos(x)}\right)$

$= \dfrac{1+1}{0} = \underline{\underline{\infty}}.$

The limit does not exist.

(b) $\lim_{x\to 1} \left(\dfrac{1}{\ln(x)} - \dfrac{1}{x-1}\right)$

$= \lim_{x\to 1} \left(\dfrac{x-1-\ln(x)}{(x-1)\ln(x)}\right)$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 1} \left(\dfrac{1-\frac{1}{x}}{\ln(x)+\frac{x-1}{x}}\right)$

$= \lim_{x\to 1} \left(\dfrac{x-1}{x\ln(x)+x-1}\right)$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 1} \left(\dfrac{1}{\ln(x)+\frac{x}{x}+1}\right)$

$= \lim_{x\to 1} \left(\dfrac{1}{\ln(x)+2}\right)$

$= \underline{\underline{\dfrac{1}{2}}}.$

(c) $\lim_{x \to 1} \left(\frac{\ln(x)}{x^2 - x} \right)$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule} \right]$

$= \lim_{x \to 1} \left(\frac{1/x}{2x - 1} \right)$

$= \lim_{x \to 1} \left(\frac{1}{2x^2 - x} \right) = \underline{\underline{1}}.$

6. L'Hospital's rule has been applied in the first step, viz. $\lim_{x \to 0} \frac{\cos(x)}{x^2} = \lim_{x \to 0} \frac{-\sin(x)}{2x}$, incorrectly. $\lim_{x \to 0} \frac{\cos(x)}{x^2} = \frac{1}{0} (= \infty)$, which is not of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, i.e. an indeterminate form.

7. (a) $\lim_{x \to \infty} \left(\frac{1}{x} e^x \right)$ $\left[= \frac{\infty}{\infty} \Rightarrow \text{L'Hospital's rule} \right]$

$= \lim_{x \to \infty} \left(\frac{e^x}{1} \right) = \underline{\underline{\infty}}.$ <u>The limit is undefined.</u>

(b) $\lim_{x \to \infty} \frac{x^2}{e^x}$ $\left[= \frac{\infty}{\infty} \Rightarrow \text{L'Hospital's rule} \right]$

$= \lim_{x \to \infty} \frac{2x}{e^x}$ $\left[= \frac{\infty}{\infty} \Rightarrow \text{L'Hospital's rule} \right]$

$= \lim_{x \to \infty} \frac{2}{e^x} = \frac{2}{\infty} = \underline{\underline{0}}.$

(c) $\lim_{x \to 0^+} \sin(x) \ln(x)$. This is of the form $0 \times \infty$ so we must manipulate.

$$\lim_{x \to 0^+} [\sin(x) \ln(x)]$$

$$= \lim_{x \to 0^+} \frac{\ln(x)}{1/\sin(x)} \qquad \left[= \frac{-\infty}{\infty} \Rightarrow \text{L'Hospital's rule}\right]$$

$$= \lim_{x \to 0^+} \frac{1/x}{-\frac{\cos(x)}{\sin(x)}}$$

$$= \lim_{x \to 0^+} \left(-\frac{\sin^2(x)}{x\cos(x)}\right) \qquad \left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$$

$$= \lim_{x \to 0^+} \left(-\frac{2\sin(x)\cos(x)}{\cos(x) - x\sin(x)}\right) = \frac{0}{1} = \underline{\underline{0.}}$$

8. (a) $$\lim_{x \to 0} \frac{x - \sin(x)}{x^2 e^x} \qquad \left[= \frac{0-0}{0 \times 1} = \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$$

$$= \lim_{x \to 0} \frac{1 - \cos(x)}{2xe^x + x^2e^x} \qquad \left[= \frac{1-1}{0+0} = \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$$

$$= \lim_{x \to 0} \frac{-\sin(x)}{2e^x + 2xe^x + 2xe^x + x^2e^x}$$

$$= \lim_{x \to 0} \frac{-\sin(x)}{e^x(2 + 4x + x^2)} = \frac{0}{1(2+0+0)} = \underline{\underline{0.}}$$

(b) $\lim_{x \to 0} \frac{1-\cos(x)}{\sin^2(x)}$ $\left[= \frac{1-1}{0} = \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x \to 0} \frac{\sin(x)}{2\sin(x)\cos(x)}$

$= \lim_{x \to 0} \frac{1}{2\cos(x)} = \frac{1}{2}$.

(c) $\lim_{x \to 1} \left(\frac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6} \right)$

$= \lim_{x \to 1} \left(\frac{(x-1)(x^3 - 6x^2 + 2x + 2)}{(x-1)(x+6)} \right)$

$= \lim_{x \to 1} \left(\frac{x^3 - 6x^2 + 2x + 2}{x+6} \right) = \frac{1-6+2+2}{1+6} = -\frac{1}{7}$.

[Note: $\lim_{x \to 1} \left(\frac{x^4 - 7x^3 + 8x^2 - 2}{x^3 + 5x - 6} \right) = \frac{0}{0}$, so we can, alternatively, use L'Hospital's rule.]

(d) $\lim_{x \to 0} \left(\operatorname{cosec}(x) - \frac{1}{x} \right)$

$= \lim_{x \to 0} \left(\frac{x - \sin(x)}{x\sin(x)} \right)$ $\left[= \frac{0-0}{0} = \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x \to 0} \left(\frac{1-\cos(x)}{\sin(x) + x\cos(x)} \right) = \left[\frac{1-1}{0+0} = \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x \to 0} \left(\frac{\sin(x)}{2\cos(x) - x\sin(x)} \right)$

$= \frac{0}{2-0} = 0$.

(e) $\lim_{x\to 0} x^2\ln(x) = 0 \times -\infty$ so we must manipulate.

$= \lim_{x\to 0} \frac{\ln(x)}{1/x^2}$ $\left[= \frac{-\infty}{\infty} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 0} \frac{1/x}{-2/x^3} = \lim_{x\to 0}\left(-\frac{1}{x}\times\frac{x^3}{2}\right) = \lim_{x\to 0}\left(-\frac{x^2}{2}\right) = \underline{\underline{0.}}$

(f) $\lim_{x\to 0}\left(\frac{e^x + e^{-x} - x^2 - 2}{\sin^2(x) - x^2}\right) = \frac{1-1-0-2}{0-0} = \infty.$

$\underline{\underline{\text{The limit does not exist.}}}$

(g) $\lim_{x\to 0} \frac{\cot(x)}{\cot(2x)} = \lim_{x\to 0}\left(\frac{\cos(x)}{\sin(x)} \times \frac{\sin(2x)}{\cos(2x)}\right)$

$= \lim_{x\to 0}\left(\frac{2\cos^2(x)}{\cos(2x)}\right) = \underline{\underline{2.}}$

(h) $\lim_{x\to \infty}\left(\frac{5x + 2\ln(x)}{x + 3\ln(x)}\right)$ $\left[= \frac{\infty}{\infty} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to\infty}\left(\frac{5 + 2/x}{1 + 3/x}\right) = \underline{\underline{5.}}$

(i) $\lim_{x\to 0}\left(\frac{\cos(2x) - \cos(x)}{\sin^2(x)}\right)$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 0}\left(\frac{-2\sin(2x) + \sin(x)}{2\sin(x)\cos(x)}\right)$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 0}\left(\frac{-4\cos(2x) + \cos(x)}{2\cos(2x)}\right)$ (since $2\sin(x)\cos(x) = \sin(2x)$)

$= \frac{-4+1}{2} = \underline{\underline{-3/2.}}$

9. (a) i. $\lim_{x\to 8}\left(\frac{x-8}{\sqrt[3]{x}-2}\right)$ $\left[=\frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 8}\left(\frac{1}{\frac{1}{3}x^{-2/3}-2}\right)$

$= \lim_{x\to 8}\left(\frac{1}{\frac{1}{3x^{2/3}}-2}\right)$

$= \lim_{x\to 8}\left(\frac{3\sqrt[3]{x^2}}{1-6\sqrt[3]{x^2}}\right) = \frac{3\times 4}{1-6\times 4} = -\frac{12}{23}.$

ii. $\lim_{x\to 1}\left(\frac{e^x-e}{x-1}\right)$ $\left[=\frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 1}\left(\frac{e^x}{1}\right) = e.$

(b) Given $f(\pi)=0$ and $\lim_{x\to\pi}\frac{f(x)}{\sin(x)}=2$

Now $\lim_{x\to\pi}\frac{f(x)}{\sin(x)} = \frac{f(\pi)}{\sin(\pi)} = \frac{0}{0}$ which is indeterminate so we use L'Hospital's rule.

$\lim_{x\to\pi}\frac{f(x)}{\sin(x)} = \lim_{x\to\pi}\frac{f'(x)}{\cos(x)} = \frac{f'(\pi)}{-1} = -f'(\pi).$

But $\lim_{x\to\pi}\frac{f(x)}{\sin(x)} = 2$, so $f'(\pi) = -2$.

10. $$\lim_{x\to 0} x^{\sin(x)} = \lim_{x\to 0} e^{\sin(x)\ln(x)}$$

$$= \exp\left(\lim_{x\to 0} \frac{\ln(x)}{\frac{1}{\sin(x)}}\right)$$

Now $$\lim_{x\to 0} \frac{\ln(x)}{1/\sin(x)} \quad \left[= \frac{-\infty}{\infty} \Rightarrow \text{L'Hospital's rule}\right]$$

$$= \lim_{x\to 0} \frac{1/x}{-\cos(x)/\sin^2(x)}$$

$$= \lim_{x\to 0} \left(-\frac{\sin^2(x)}{x\cos(x)}\right) \quad \left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$$

$$= \lim_{x\to 0} \left(-\frac{2\sin(x)\cos(x)}{\cos(x) - x\sin(x)}\right) = 0.$$

$$\therefore \lim_{x\to 0} x^{\sin(x)} = e^0 = \underline{\underline{1.}}$$

11. $$(1+x)^{1/x} = e^{\ln(1+x)^{1/x}} = e^{\frac{1}{x}\ln(1+x)}$$

Now $$\lim_{x\to 0^+} \left(\frac{1}{x}\ln(1+x)\right) \quad \left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$$

$$= \lim_{x\to 0^+} \left(\frac{1}{1+x}\right) = 1$$

$$\therefore \lim_{x\to 0^+} (1+x)^{\frac{1}{x}} = e^1 = \underline{\underline{e.}}$$

12. $\lim_{x\to\infty} x\left(a^{1/x}-1\right)$

$= \lim_{x\to\infty} \frac{a^{1/x}-1}{1/x}$ $\quad \left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

Now $\frac{d}{dx}\left(a^{1/x}\right) = \frac{d}{dx}\left(e^{\frac{1}{x}\ln(a)}\right)$

$= e^{\frac{1}{x}\ln(a)}\left(-\frac{1}{x^2}\ln(a)\right)$

$= -a^{1/x}\cdot\frac{\ln(a)}{x^2}$

So $\lim_{x\to\infty}\frac{a^{1/x}-1}{1/x} = \lim_{x\to\infty}\frac{-a^{1/x}\ln(a)}{x^2}\cdot\frac{1}{-1/x^2}$

$= \lim_{x\to\infty} a^{1/x}\ln(a) = 1.\ln(a) = \underline{\underline{1.}}$

13. (a) $\lim_{x\to 1^+} x^{\frac{1}{x-1}}$

$= \lim_{x\to 1^+} e^{\frac{1}{x-1}\ln(x)} \quad = e^{\lim_{x\to 1^+}\frac{\ln(x)}{x-1}}$

Now $\lim_{x\to 1^+}\frac{\ln(x)}{x}$ $\quad \left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 1^+}\frac{1}{x} = 1.$

$\therefore \lim_{x\to 1^+} x^{\frac{1}{x+1}} = e^1 = \underline{\underline{e.}}$

(b) $\lim_{x\to 0} (\sin(x))^x = \lim_{x\to 0} e^{x\ln(\sin(x))} = e^{\lim_{x\to 0} x\ln(\sin(x))}$

Now $\lim_{x\to 0} x\ln(\sin(x))$ (of the form $0\times\infty$)

$= \lim_{x\to 0} \dfrac{\ln(\sin(x))}{1/x}$ $\left[= \dfrac{-\infty}{\infty} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 0} \dfrac{\cos(x)/\sin(x)}{-1/x^2}$

$= \lim_{x\to 0} \left(-\dfrac{x^2\cos(x)}{\sin(x)}\right)$ $\left[= \dfrac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 0} \left(-\dfrac{2x\cos(x) - x^2\sin(x)}{\cos(x)}\right) = 0.$

So $\lim_{x\to 0} (\sin(x))^x = e^0 = \underline{\underline{1}}.$

(c) $\lim_{x\to \infty} (x+1)^{2/x} = \lim_{x\to\infty} e^{\frac{2}{x}\ln(x+1)} = e^{\lim_{x\to\infty} \frac{2}{x}\ln(x+1)}$

Now $\lim_{x\to\infty} \dfrac{2\ln(x+1)}{x}$ $\left[= \dfrac{\infty}{\infty} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to\infty} \dfrac{2}{x+1} = 0$

$\therefore \lim_{x\to\infty} (x+1)^{2/x} = e^0 = \underline{\underline{1}}.$

14. (a) $\lim_{x\to 0}(\cos(x))^{1/x} = \lim_{x\to 0} e^{\frac{1}{x}\ln(\cos(x))}$

$= e^{\lim_{x\to 0}\frac{1}{x}\ln(\cos(x))}$

Now $\lim_{x\to 0}\frac{\ln(\cos(x))}{x}$ $\left[=\frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 0}\frac{-\frac{\sin(x)}{\cos(x)}}{1} = \lim_{x\to 0}(-\tan(x)) = 0.$

So $\lim_{x\to 0}(\cos(x))^{1/x} = e^0 = \underline{\underline{1.}}$

(b) $\lim_{x\to 0}\left(\frac{1}{x}\right)^x = \lim_{x\to 0} e^{-x\log x}$

$= e^{\lim_{x\to 0}(-x\log x)}$

Now, $\lim_{x\to \infty}(-x\log x)$ $(0\times\infty)$

$= -\lim_{x\to 0}\left(\frac{\log(x)}{1/x}\right)$ $\left[=\frac{-\infty}{\infty} \Rightarrow \text{L'Hospital's rule}\right]$

$= -\lim_{x\to 0}\left(\frac{1/x}{-1/x^2}\right)$

$= \lim_{x\to 0}(x) = 0.$

$\therefore \lim_{x\to 0}\left(\frac{1}{x}\right)^x = e^0 = \underline{\underline{1.}}$

15. (a) $f(1) = 1$ and $f'(1) = e$.

As $x \to 1$, $\dfrac{[f(x)]^4 - 1}{x^2 - 1} \to \dfrac{1-1}{1-1} = \dfrac{0}{0} \Rightarrow$ L'Hospital's rule.

$$\lim_{x \to 1} \frac{[f(x)]^4 - 1}{x^2 - 1} = \lim_{x \to 1} \frac{4[f(x)]^3 \cdot f'(x)}{2x}$$

$$= \frac{4 \times [1]^3 \times e}{2} = \underline{\underline{2e}}.$$

(b) $\displaystyle\lim_{x \to 0} \frac{f(x)}{e^x - 1} = e$ and $f(0) = 0$.

As $x \to 0$, $\dfrac{f(x)}{e^x - 1} \to \dfrac{f(0)}{e^0 - 1} = \dfrac{0}{0} \Rightarrow$ L'Hospital's rule.

$$\lim_{x \to 0} \frac{f(x)}{e^x - 1} = \lim_{x \to 0} \frac{f'(x)}{e^x} = \frac{f'(0)}{e^0} = f'(0).$$

But $\displaystyle\lim_{x \to 0} \frac{f(x)}{e^x - 1} = e$, $\Rightarrow$ $\underline{\underline{f'(0) = e}}$.

16. (a) $\lim_{x\to 0} \frac{\sin(\alpha x)}{\sin(\beta x)}$ $\left[= \frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 0} \frac{\alpha \cos(\alpha x)}{\beta \cos(\beta x)} = \underline{\underline{\alpha/\beta}}.$

(b) $\lim_{x\to 0} \frac{\sin(x^\alpha)}{(\sin(x))^\alpha}$, $\alpha \in \mathbb{Z}^+$. $\beta \in \mathbb{Z}^+$.

The limit $= \frac{0}{0}$ so use L'Hospital's rule.

$\lim_{x\to 0} \frac{\sin(x^\alpha)}{(\sin(x))^\alpha}$

$= \lim_{x\to 0} \frac{\alpha x^{\alpha-1} \cos(x^\alpha)}{\alpha(\sin(x))^{\alpha-1} \cos(x)}$

$= \lim_{x\to 0} \left(\frac{x}{\sin(x)}\right)^{\alpha-1} \cdot \frac{\cos(x^\alpha)}{\cos(x)}$

$= (1)^{\alpha-1} \cdot \frac{1}{1}$

$= \underline{\underline{1}}.$

17. (a) $\lim_{x\to 0} \dfrac{\cos(\alpha x)-\cos(\beta x)}{x^2}$ $\left[=\frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 0} \dfrac{-\alpha\sin(\alpha x)+\beta\sin(\beta x)}{2x}$ $\left[=\frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to 0} \dfrac{-\alpha^2\cos(\alpha x)+\beta^2\cos(\beta x)}{2} = \dfrac{1}{2}\left(\beta^2-\alpha^2\right).$

(b) $\lim_{x\to \beta} \dfrac{\sin^2(x)-\sin^2(\beta)}{x^2-\beta^2}$ $\left[=\frac{0}{0} \Rightarrow \text{L'Hospital's rule}\right]$

$= \lim_{x\to \beta} \dfrac{2\sin(x)\cos(x)}{2x} = \dfrac{1}{2\beta}\sin(2\beta).$

Exercise E.9.1

1 **a** $-2x$ **b** $-\frac{x}{y}$ **c** $\frac{1}{x^3y}$ **d** $-\frac{y}{x+1}$

e $-\frac{ye^x}{1+e^x}$ **f** $\frac{\sin x - y}{x}$ **g** $-x$ **h** $\frac{1-3x^4y}{x^5}$

i $-\frac{y\cos x + 2}{\sin x}$ **j** -1 **k** $\frac{4x^3}{3y^2+1}$ **l** $\sqrt{x+y}-1$

2 (1,5), 0

4 $\left(\frac{3-2\sqrt{10.6}}{2}, \frac{80+4\sqrt{265}}{40}\right), \left(\frac{3+2\sqrt{10.6}}{2}, \frac{80-4\sqrt{265}}{40}\right)$

5 **a** $y = \frac{x \pm \sqrt{5x^2-80}}{2}$ **c** $\frac{dy}{dx} = \frac{2x+y}{2y-x}$ **d** $\frac{5x \pm \sqrt{5x^2-80}}{2\sqrt{5x^2-80}}$

e Hyperbola

6 **a** Dom = Ran = [−2,2] **b** $-\frac{x^3}{y^3}$ **c** $-\frac{x^3}{(\sqrt[4]{16-x^4})^3}$ **d** small

e Dom = Ran = $[-k,k]$ **f** $\frac{dy}{dx} = -\frac{x^{2n-1}}{y^{2n-1}}$

7 **a** $\frac{-\nu}{p\gamma}$ **b** $\frac{n(m-1)x^{m-2}}{m(n+1)y^n}$

8 **a** $\frac{1}{11}$ **b** -1

9 **a** $\frac{y}{xy-x}$ **b** $\frac{(1+y^2)(\tan^{-1}y - 1)}{1-x+y^2}$

10 **a** undefined **b** At (0.8042, 0.5), grad = 1.32; at (0.0646, 0.5), grad = 3.74

Exercise E.9.2

1 **a** $4\pi r$ cm²s⁻¹ **b** 4π cms⁻¹

2 6 cm²s⁻¹

3 **a** $\frac{dA}{dt} = -\frac{3}{2}\sqrt{2}x$ cm²s⁻¹ (x = side length) **b** $-\frac{3}{2}\sqrt{2}$ cms⁻¹

4 **a** 37.5 cm³h⁻¹ **b** 30 cm²h⁻¹ **c** 0.96 g⁻¹cm³h⁻¹

5 ~ 0.37 cms^{-1}

6 –0.24 cm^3min^{-1}

7 **a** 0.035 ms^{-1} **b** 0.035 ms^{-1}

8 8π cm^3min^{-1}

9 854 kmh^{-1}

10 $\frac{53}{6}$

11 2 rad s^{-1}

12 **a** $V = h^2 + 8h$ **b** $\frac{4}{15}$ m min^{-1} **c** 0.56 m^2min^{-1}

13 $\frac{3\sqrt{10}}{200}$ m min^{-1}

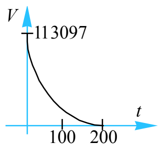
14 $10\sqrt{2}$ cm^3s^{-1}

15 0.9 ms^{-1}

16 –3.92 ms^{-1}

17 **a** $x = 30 - 0.15t$ **b** [0, 200] **c i** 1531 cm^3s^{-1} **i** **i** 15.90 cm^2s^{-1}

d



18 ~1.24 ms^{-1}

19 ~0.0696 ms^{-1}

20 **a** $y = \sqrt{119 + 20t - 4t^2}$ **b** ~0.516 ms^{-1}

21 **a** 0.095 cms^{-1} **b** 0.6747 cm^2s^{-1}

22 **a i** $x = 70t$ **ii** $y = 80t$ **b** $130t$ **c** 130 kmh^{-1}

d 14.66 kmh^{-1}

23 –0.77 ms^{-1}

24 0.40 ms^{-1}

25 3.2 ms^{-1}

26 0.075 m min^{-1}

27 1.26° per sec

28 $\frac{5}{2564} \approx 0.002$ rad per second

29 **a** 9% per second **b** 6% per second

30 0.064

31 8211 per year

32 4% per second

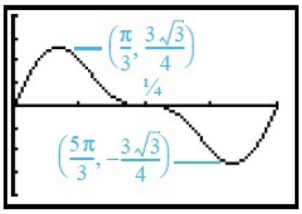
33 −0.25 rad per second

Exercise E.9.3

1. 22.6 m
2. a 1.5 mh^{-1} b \$19.55 per km
3. a 400 b \$46400000
4. \$273.86
5. \$0.40
6. 1.97 m
7. 0.45 m^3
8. 5 m by 5 m
9. 128

10 $r = \frac{50}{4+\pi} \approx 7.00$, dim of rect. aprrox 7.00 m by 7.00 m.

11. $\theta = \frac{\pi}{6}$

12 a 

b $\frac{3\sqrt{3}}{2}$ units **c** At infl. pts. when $\cos x = -\frac{1}{4}$.

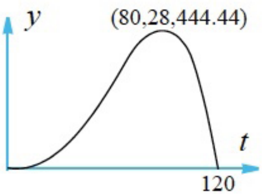
13 648 m^2

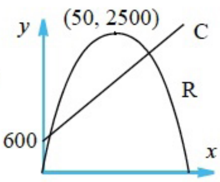
14 a 10.5 **b** 5.25

15 72

16 a $y = 100 - 2x$ **b** $A = x(100 - 2x), 0 < x < 50$ **c** $x = 25, y = 50$

17 a $\frac{100}{x} - \frac{1}{2}x, 0 < x < 10\sqrt{2}$ **b** $\frac{2000}{9}\sqrt{6} \approx 544.3$ cm^3

18 a 400 mLs^{-1} **b** 40 s **c** 

19 a 

b 8.38, 71.62 **c** $9 \le x \le 71$ **d** $80x - x^2 - 600$, \$1000

20 $\left(\sqrt{\frac{11}{2}}, \frac{7}{2}\right)$ & $\left(-\sqrt{\frac{11}{2}}, \frac{7}{2}\right)$

21 $5\sqrt{2}$ by $\frac{5}{2}\sqrt{2}$

22 4 by $\frac{8}{3}$

23 $348 - 8\sqrt{170} \sim 243.7$ cm^2

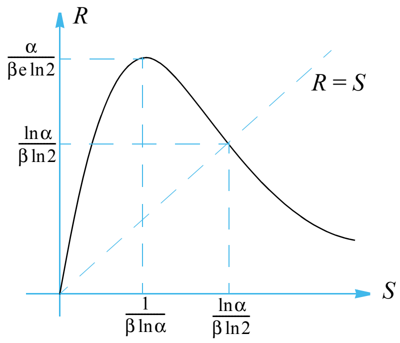
24 2

25 radius = $\sqrt{\frac{10}{3}}$ cm, height = $2\sqrt{\frac{10}{3}}$ cm

26 $\sqrt[3]{\frac{15}{\pi}}$

27 5 cm

28 **a** $h = \dfrac{24r^2}{r^2 - 144}$ **b** $\dfrac{8\pi r^4}{r^2 - 144}$ **c** $r = 12\sqrt{2}$, $h = 48$
29 $r : h = 1 : 2$
30 ~ (0.55, 1.31)
31 **b** 2.5 m
32 altitude = $\frac{1}{3}$ height of cone
33 ~ 1.640 m wide and 1.040 m high
34 $\dfrac{2\sqrt{2}}{\sqrt{3}}\pi$
35 where XP : PY = $b : a$
36 5 km
37 $r : h = 1 :1$
38 $\frac{4}{3}$ cm
39 2 : 1
40 $\dfrac{10}{\sqrt{3\pi}}$
41 0.873 km from P
42 **b** $r = 3\sqrt{2}$, $h = 6\sqrt{2}$
43 **b** when $\theta = \arcsin\left(\frac{5}{6}\right)$, i.e. approx. 6.030 km from P.
44 **a** $\tan\theta = \dfrac{xl}{x^2 + k(l + k)}$ **b** $x = \sqrt{k^2 + kl}$
45 **c** if $k < c$, swimmer should row directly to Q.
46 **a** **i** $\pi r^2 h + \frac{2}{3}\pi r^3$ **ii** $3\pi r^2 + 2\pi rh$ **c** $r : h = 1:1$
47 $(a^{2/3} + b^{2/3})^{3/2}$
48 **b** 4 km along the beach **c** row directly to destination
49 **a**



b First integer greater than $\dfrac{\ln\alpha}{\beta\ln 2}$

50 $\sqrt{295 \times 145} \approx 207$ m
51 **a** isosceles triangle **b** isoscles right-angled triangle
52 $r^2\left(\frac{1}{2} + \frac{1}{4}\sqrt{3}\right)$ sq. units
53 $\dfrac{4k^2 + 4k - 1}{8k(k + 1)}$ sq. units
54 **a** $c = -r$

Exercise E.10.1

1

a $\frac{2}{3}(5x^2+2)^{3/2}+c$ **b** $-\frac{1}{3(x^3+4)}+c$ **c** $\frac{3}{8}(1-2x^2)^4+c$

d $\frac{1}{5}(9+2x^{3/2})^5+c$ **e** $\frac{9}{4}(x^2+4)^{4/3}+c$ **f** $\frac{-1}{2(x^2+3x+1)^2}+c$

g $4\sqrt{x^2+2}+c$ **h** $\frac{1}{12(1-x^4)^3}+c$ **i** $\frac{2}{3}(1+e^{3x})^{3/2}+c$

j $\frac{-1}{2(x^2+2x-1)}+c$ **k** $\frac{2}{3}\sqrt{x^3+3x+1}+c$ **l** $\frac{1}{12}(3+4x^2)^{3/2}+c$

m $2\sqrt{e^x+2}+c$ **n** $-\frac{1}{4}(1-e^{-2x})^{-2}+c$ **o** $\frac{2}{3}(x^3+1)^5+c$

p $\frac{1}{24}(x^4+8x-3)^6+c$ **q** $\frac{1}{5}(x^4+5)^{5/2}+c$ **r** $-\sqrt{1-\sin 2x}+c$

s $\frac{2}{9}(4+3\sin x)^{3/2}+c$ **t** $-\frac{1}{12(1+3\tan 4x)}+c$ **u** $\frac{3}{2}(x+\cos x)^{2/3}+c$

v $-\frac{1}{2}\cos^4\frac{x}{2}+c$ **w** $2\sqrt{1+x\sin x}+c$ **x** $\frac{4}{3}(x^{1/2}+1)^{3/2}+c$

2

a $e^{x^2+1}+c$ **b** $6e^{\sqrt{x}}+c$ **c** $\frac{1}{3}e^{\tan 3x}+c$

d $-e^{-(ax^2+bx)}+c$ **e** $-6e^{\cos\frac{x}{2}}+c$ **f** $-4e^{(4+x^{-1})}+c$

g $-\frac{1}{2}\cos(2e^x)+c$ **h** $\frac{1}{2(1-e^{2x})}+c$ **i** $-\ln(1+e^{-x})+c$

j $\frac{5}{2}\ln(1+2e^x)+c$ **k** $-\frac{2}{3a}(4+e^{-ax})^{3/2}+c$ **l** $\frac{(\ln(1+e^{2x}))^2}{4}+c$

3

a $-\cos(x^2+1)+c$ **b** $-10\cos\sqrt{x}+c$ **c** $-2\sin\left(2+\frac{1}{x}\right)+c$

d $-\frac{2}{3}(\cos x)^{3/2}+c$ **e** $-\frac{1}{3}\log(\cos 3x)+c$ **f** $\frac{4}{3}\log(1+\tan 3x)+c$

g $\frac{-4}{3(\tan(3x)+1)}+c$ **h** $2\sin(\ln x)+c$ **i** $-\frac{1}{6}(1+\cos 2x)^{3/2}+c$

j $\sin(e^x)+c$ **k** $-e^{(-x^3+2)}+c$ **l** $\left[\ln\left(\sin\frac{1}{2}x\right)\right]^2+c$

m $\sec x+c$ **n** $\frac{1}{4}[\ln(1+2e^x)]^2+c$ **o** $\tan\left(\frac{1}{3}x^3-3x\right)+c$

4

a $\text{Tan}^{-1}\left(\frac{x}{2}\right)+c$ **b** $\text{Tan}^{-1}\left(\frac{x}{3}\right)+c$ **c** $\text{Tan}^{-1}\left(\frac{x}{\sqrt{5}}\right)+c$

d $\text{Sin}^{-1}\left(\frac{x}{5}\right)+c$ **e** $\text{Sin}^{-1}\left(\frac{x}{4}\right)+c$ **f** $\text{Cos}^{-1}\left(\frac{x}{3}\right)+c$

5

a $3\text{Tan}^{-1}x+c$ **b** $5\text{Sin}^{-1}x+c$ **c** $\text{Sin}^{-1}\left(\frac{x}{2}\right)+c$

d $\text{Sin}^{-1}\left(\frac{x}{3}\right)+c$ **e** $\frac{1}{2}\text{Sin}^{-1}2x+c$ **f** $\frac{1}{2}\text{Sin}^{-1}\left(\frac{2x}{3}\right)+c$

g $\frac{1}{5}\text{Sin}^{-1}\left(\frac{5x}{2}\right)+c$ **h** $\text{Tan}^{-1}2x$ **i** $\frac{1}{6}\text{Tan}^{-1}\left(\frac{2x}{3}\right)+c$

j $\frac{1}{12}\text{Tan}^{-1}\left(\frac{4x}{3}\right)+c$ **k** $\frac{\sqrt{5}}{15}\text{Tan}^{-1}\left(\frac{\sqrt{5}x}{3}\right)+c$ **l** $\frac{1}{\sqrt{5}}\text{Sin}^{-1}\left(\sqrt{\frac{5}{3}}x\right)+c$

6

a $\frac{531377}{9}$ **b** $-2\sqrt{2}+2\sqrt{1+e}$ **c** $3\ln 2(2+\sqrt{2})$

d $4\text{Tan}^{-1}\left(\frac{\pi}{2}\right)$ **e** $\sin e-\sin(e^{-1})$ **f** $\frac{2}{3}\left[1-\cos\left(\frac{\pi}{2}\right)^{3/2}\right]$

g $\frac{2}{3}$ **h** $e-e^{-1}$ **i** $\ln 2$

j	$\frac{7\sqrt{7}}{3}$	**k**	0	**l**	$\frac{3}{5}$
m	$\frac{\pi}{4}$	**n**	$\frac{\pi}{2}-\text{Tan}^{-1}(2)$	**o**	$\frac{1}{3}\text{Tan}^{-1}9$
p	$\frac{1}{2}\text{Sin}^{-1}\left(\frac{2}{3}\right)$	**q**	$\frac{1}{4}\left(\pi-2\text{Sin}^{-1}\left(\frac{2}{3}\right)\right)$	**r**	$\frac{1}{3}\text{Tan}^{-1}\left(\frac{3}{2}\right)$
s	$\frac{1}{6}\left(\text{Tan}^{-1}\left(\frac{3}{4}\right)-\text{Tan}^{-1}\left(\frac{3}{16}\right)\right)$	**t**	$\frac{1}{64}$	**u**	$\frac{2}{3}-\frac{\sqrt{2}}{3}$
v	$\frac{1}{3}$	**w**	$\frac{3\pi}{4}$	**x**	$-\frac{1}{60}$

Exercise E.10.2

1.
a	$\arctan(2x)$	b	$\arcsin\left(\frac{x}{3}\right)$	c	$\text{Cos}^{-1}(2x)$
d	$\text{Sin}^{-1}(4x)$	e	$\arctan\left(\frac{x}{2}\right)$	f	$\arcsin(x-1)$
g	$\arccos\left(\frac{x}{4}\right)$	h	$\arcsin\left(\frac{x+1}{2}\right)$	i	$\text{Tan}^{-1}(x-4)$
j	$\arcsin\left(\frac{2-x}{2}\right)$	k	$\arctan\left(\frac{2x}{3}\right)$	l	$\arccos\left(\frac{2x-1}{3}\right)$

2.
a	$\pi/4$	b	$\pi/6$	c	$\frac{1}{3}\tan^{-1}\left(\frac{2}{3}\right)$
d	$\frac{2}{3}\arcsin\left(\frac{\sqrt{3}}{3}\right)$	e	π	f	$\pi/4$
g	$\frac{3e^{-2}}{2}+\frac{\pi}{4}-\frac{1}{2}$				

3.
a	$(\ln 4)4^x$	**b**	$(\ln 3)3^x$	**c**	$(\ln 8)8^x$	**d**	$3(\ln 5)5^x$
e	$7(\ln 6)6^x$	**f**	$2(\ln 10)10^x$	**g**	$(\ln 6)6^{x-2}$	**h**	$3(\ln 2)2^{3x+1}$
i	$-5(\ln 7)7^{3-x}$						

4.
a	$\frac{2^x}{\log_e 2}$	b	$\frac{3^{2x}}{2.\log_e 3}+c$	c	$\frac{4^x}{32.\log_e 2}+c$
d	$\frac{81^x}{972.\log_e 3}+c$	e	$\frac{2.3^{x/2}}{\log_e 3}+c$	f	$\frac{4.3^{3x/4}}{3.\log_e 3}+c$

5. a $\frac{7}{\log_e 2}$ b $\frac{80}{3.\log_e 3}$ c $\frac{3280}{9.\log_e 3}$

d $\frac{31}{\log_e 2}$ e $\frac{2.(\log_e 2+84)}{\log_e 2}$

6. 3

7. 5

8. 1.

Exercise E.10.3

1. a $\frac{1}{4}\log_e\left(\frac{x-1}{x+3}\right)$ b $\frac{1}{6}\log_e\left(\frac{x-2}{x+4}\right)$ c $\frac{1}{7}\log_e\left(\frac{2x-1}{x+3}\right)$

d $\frac{1}{17}\log_e\left(\frac{2x-5}{3x+1}\right)$ e $\frac{-3}{55}\log_e(3x-1)+\frac{2}{45}\log_e(2x+1)+\frac{1}{99}\log_e(x-4)$

f $\frac{x^3}{3}+\frac{x^2}{2}+x+\log_e(x-1)$ g $\frac{x^2}{2}+\frac{1}{2}\log_e\left(x^2-1\right)$

h $-\frac{x^2}{2}-\frac{1}{2}\log_e\left(x^2+1\right)$ i $2x+\frac{1}{2}\log_e\left(\frac{(x-4)^2}{(x+2)^2(2x-1)^3}\right)$

j $\frac{1}{4}\log_e(x+1)+\frac{27}{4}\log_e(x-3)+\frac{x^2}{2}+2x$

2. a $\log_e\frac{3}{2}$ b $\frac{1}{11}\log_e\frac{3}{25}$ c $\frac{1}{60}\log_e 6912$

d $\frac{17}{12}\log_e(5)-\frac{8}{3}\log_e(2)+1$

e $\frac{1}{42}\log_e(5)-\frac{19}{35}\log_e(3)-\frac{1}{15}\log_e(2)$

f $-\frac{5}{2}\arctan\left(\frac{2}{3}\right)-\frac{5}{2}\arctan\left(\frac{1}{2}\right)+\frac{5\pi}{4}$

2. 2

3. 2

5. $-2.\log_e\left(\frac{15}{16}\right)$

6. $\frac{3}{4}-\log_e(2)$

7. 3

8. 0.4990

Exercise E.10.4

1

a $\frac{2}{3}(x^2+1)^{3/2}+c$

b $\frac{2}{3}(x^3+1)^{3/2}+c$

c $-\frac{1}{3}(4-x^4)^{1.5}+c$

d $\ln(x^3+1)+c$

e $-\frac{1}{18(3x^2+9)^3}+c$

f $e^{(x^2+4)}+c$

g $\ln(z^2+4z-5)+c$

h $-\frac{3}{8}(2-t^2)^{4/3}+c$

i $e^{\sin x}+c$

j $\ln[e^x+1]+c$

k $\frac{1}{5}\sin^5 x+c$

l $\frac{2}{5}(x+1)^{5/2}-\frac{2}{3}(x+1)^{3/2}+c$

2

a $\frac{1}{10}(2x-1)^{5/2}+\frac{1}{6}(2x-1)^{3/2}+c$

b $-\frac{2}{3}(1-x)^{3/2}+\frac{4}{5}(1-x)^{5/2}-\frac{2}{7}(1-x)^{7/2}+c$

c $\frac{2}{5}(x-1)^{5/2}+\frac{4}{3}(x-1)^{3/2}+c$

d $e^{\tan x}+c$

e $-\ln(1-2x^2)+c$

f $\frac{1}{1-2x^2}+c$

g $\frac{1}{2}(\ln x)^2+c$

h $-\ln(1+e^{-x})+c$

i $\ln(\ln x)+c$

3

a 0

b $\frac{2\ln 2}{3}$

c $\ln\frac{77}{54}$

d $\ln 2$

e $\frac{1}{3}\ln 2$

f $\frac{1}{4}$

g $\frac{76}{15}$

h $\frac{16}{15}$

i $\frac{2}{3}(1+e)^{3/2}(1-e^{-3/2})$

4

a	$\frac{7\sqrt{7}}{3}-\frac{8}{3}$	**b**	$\frac{3}{8}(\cos\pi^2-1)$	**c**	$\frac{1042}{5}$
d	$\ln 4$	**e**	1	**f**	$\frac{5}{4}(e^5-e^{-1})$
g	24 414	**h**	$\sqrt{3}-\sqrt{2}$	**i**	$\frac{1}{4}\ln 3$

5

a	$\frac{1}{4}$	**b**	$2-\frac{2}{3}\sqrt{3}$	**c**	$\frac{31}{80}$
d	$4-2\sqrt{2}$	**e**	$\ln 2$	**f**	$\frac{2}{3}$

6

a	$-\frac{2}{5}\sqrt{3}$	**b**	$\frac{2}{5}\sqrt{3}$	**c**	$\frac{26}{3}$
d	$-\frac{4}{3}$	**e**	$-\frac{56}{15}\sqrt{2}$	**f**	$3 + 2\ln 4$

7

a	$\text{Tan}^{-1}(x+3)+c$	**b**	$\frac{2}{\sqrt{3}}\text{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$	**c**	$\text{Sin}^{-1}\left(\frac{x-2}{\sqrt{5}}\right)+c$
d	$3\text{Sin}^{-1}\left(\frac{x+1}{3}\right)+c$	**e**	$2\sin^{-1}\left(\frac{2x-3}{\sqrt{29}}\right)+c$	**f**	$\frac{1}{2}\sin^{-1}\left(\frac{x^2}{3}\right)+c$
g	$\frac{1}{2}(\arcsin x)^2+c$	**h**	$-\frac{1}{3}(\arccos x)^3+c$	**i**	$-\frac{1}{2}(\arcsin x)^{-2}+c$

8 **a** $A = 1, B = -2$

9 **a** $\text{Tan}^{-1}k$ **b i** $\frac{\pi}{6}$ **ii** $\frac{\pi}{4}$ **c** $\frac{\pi}{2}, \pi$

10 $2\sqrt{x}-2\ln(\sqrt{x}+1), 2-2\ln 2$

11 $\frac{3k^2\pi}{8}$

12

a $\frac{\pi}{3}$ **b** $8\text{Sin}^{-1}\left(\frac{2}{3}\right)$ **c** $\frac{\pi}{4}$

d $\frac{1}{2}\text{Sin}^{-1}(1)$ **e** $2\sqrt{2}-2-\frac{\pi}{2}$ **f** $\frac{\pi}{4}$

g $\pi-2\text{Tan}^{-1}\left(\frac{1}{3}\right)$

Exercise E.10.5

1

a $\sin x - x\cos x + c$ **b** $4\cos\frac{x}{2}+2x\sin\frac{x}{2}+c$ **c** $2\left(4\sin\frac{x}{2}-2x\cos\frac{x}{2}\right)+c$

d $-e^{-x}(x+1)+c$ **e** $-5e^{-4x}\left(\frac{x}{4}+\frac{1}{16}\right)+c$ **f** $x\ln x - x + c$

g $\frac{x^2}{2}\ln x-\frac{x^2}{4}+c$ **h** $-\frac{1}{25}(\cos 5x+5x\sin 5x)+c$

i $12\left(x\cos\frac{x}{3}-3\sin\frac{x}{3}\right)+c$ **j** $\ln\cos x + x\tan x + c$ **k** $\frac{2}{3}x\sqrt{x}\ln x-\frac{4}{9}x\sqrt{x}+c$

2

a $\frac{2}{15}(3x-2)(x+1)^{3/2}+c$ **b** $\frac{2}{15}(3x+4)(x-2)^{3/2}+c$ **c** $\frac{2}{15}(3x+1)(x+2)^{3/2}+c$

3

a $x\text{Cos}^{-1}x-\sqrt{1-x^2}+c$ **b** $x\text{Tan}^{-1}x-\frac{1}{2}\ln(x^2+1)+c$ **c** $x\text{Sin}^{-1}x+\sqrt{1-x^2}+c$

4

a $\left(\frac{1}{2}x^2-\frac{1}{4}\right)\text{Cos}^{-1}x-\frac{1}{4}x\sqrt{1-x^2}+c$ **b** $\frac{1}{2}(x^2+1)\text{Tan}^{-1}x-\frac{x}{2}+c$

c $\frac{1}{4}(2x^2-1)\text{Sin}^{-1}x+\frac{1}{4}x\sqrt{1-x^2}+c$

5

a $\frac{1}{4}$ **b** $\frac{1}{4}(e^2+1)$ **c** $\frac{1}{4}(e^2-4)$

d $\frac{1}{4}$ **e** $\frac{4\pi-\sqrt{2}\pi-4\sqrt{2}}{32}$ **f** $\frac{1}{2}$

6 $\frac{1}{6}\ln 2+\frac{\pi}{12}-\frac{1}{6}$

7 $\frac{1}{2}[\sqrt{2}+\ln(\sqrt{2}+1)]$

8

a $\frac{x}{2}\cos(\ln x)+\frac{x}{2}\sin(\ln x)+c$

b $-\frac{x}{2}\cos(\ln x)+\frac{x}{2}\sin(\ln x)+c$

c $-\frac{1}{15}(1-x^2)(2+3x^2)\sqrt{1-x^2}+c$

Exercise E.10.6

1

a $e^x(x^2-2x+2)+c$

b $3\left(\frac{x}{2}\cos 2x+\frac{2x^2-1}{4}\sin 2x\right)+c$

c $\frac{x^4}{4}\log 2x-\frac{x^4}{16}+c$

d $-\frac{e^x}{5}(2\cos 2x-\sin 2x)+c$

e $\frac{2x}{9}\cos 3x+\frac{9x^2-2}{27}\sin 3x+c$

f $-\frac{e^{-2x}}{4}(\cos 2x-\sin 2x)+c$

g $-8\left(x^3\cos\frac{x}{2}-6x^2\sin\frac{x}{2}-24x\cos\frac{x}{2}+48\sin\frac{x}{2}\right)+c$

h $\frac{1}{2}(\ln x)^2+c$

i $2x-2x\ln(3x)+x(\ln(3x))^2+c$

j $-\frac{\cos x}{2}-\frac{\cos 3x}{6}+c$

k $\frac{1}{1+a^4}\left(a^3e^{ax}\cos\left(\frac{x}{a}\right)+ae^{ax}\sin\left(\frac{x}{a}\right)\right)+c$

l $\left(\frac{2x^3}{7}+\frac{4x^2}{35}-\frac{32x}{105}+\frac{128}{105}\right)\sqrt{x+2}+c$

m $\frac{x^4}{4}\ln ax-\frac{x^4}{16}+c$

n $2\sin^{-1}\left(\frac{x}{2}\right)-\frac{x}{2}\sqrt{4-x^2}+c$

o $\frac{3}{2}(x\sqrt{x^2-9}+9\ln(x+\sqrt{x^2-9}))+c$

p $\frac{1}{2}\ln(x^2+4)+c$

q $x-2\operatorname{Tan}^{-1}\left(\frac{x}{2}\right)+c$

2

a $\frac{\pi^2}{16}-\frac{1}{4}$

b $\frac{\pi}{8}$

c $\frac{1}{2}(e^{2\pi}-e^{\pi/2})$

d $1-\ln 2-\frac{1}{2}(\ln 2)^2$

e $\frac{a}{a^2+b^2}\left(e^{\frac{2a\pi}{b}}+e^{\frac{a\pi}{b}}\right)$

f $e-2$

Exercise E.10.7

All values are in cubic units.

1 21π

2 $p\ln 5$

3 $\frac{4}{5}\pi$

4 $\frac{\pi}{2}(e^{10} - e^{2})$

5 π^2

6 $\frac{\pi}{2}$

7 $\frac{109}{3}\pi$

8 $\pi\left(\frac{8}{3} - 2\ln 3\right)$

12 $\frac{\pi}{2}(5 - 5\sin 1)$

13 $\frac{251}{30}\pi$

14 **a** 40π **b** $\frac{242}{5}\pi$

15 **a** $\frac{8}{35}\pi$ **b** $\frac{\pi}{4}$

16 **a** $\frac{9}{2}\pi$ **b** $\frac{88}{5}\sqrt{3}\pi$

17 $\frac{3\pi}{4}$

18 $k = 1$

19 $4\pi^2 a^2$

20 $k = \frac{\pi}{2}$

21 **b i** $\frac{\pi a}{2(1 + a^2)}$ **ii** $\frac{8\pi}{15}\sqrt{\frac{a}{1 + a^2}}\left(\frac{3a^2 + 2}{1 + a^2}\right)$

22 **a** Two possible solutions: solving $a^3 - 6a^2 - 36a + 204 = 0$, a = 4.95331; solving $a^3 - 6a^2 - 36a - 28 = 0$, then a = –0.95331

b $a = \frac{100}{\pi}$

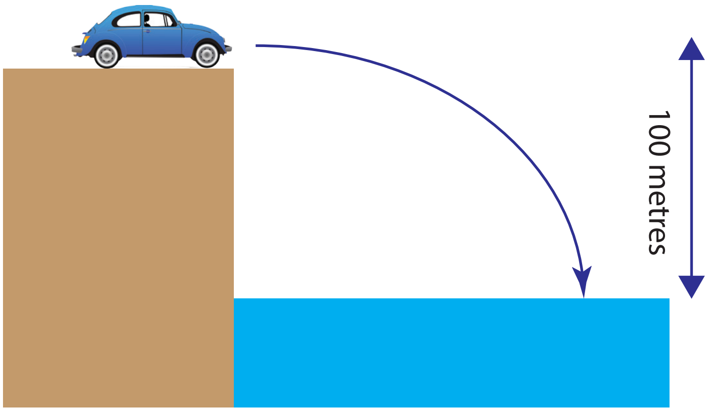
23 $\frac{28}{15}\pi$

24. **a** $\frac{1472}{15}\pi$ **b** 64π **c** $\frac{576}{5}\pi$

Exercise E.14.7

For the remaining questions, use g = 10 ms^{-2}.

6. A car travelling at 20 ms^{-1} drives off a 100 metre high cliff. How far from the base of the cliff will it hit the water?



7. Prove that $y = 3e^x - 2x - 3$ is a solution to $y'' - y' - 2 = 0, y'(0) = 1, y(0) = 0$.

8. A chemical reaction proceeds at a rate such that the amount of the principal reagent A varies with time such that $\frac{d^2A}{dt^2} = 3t$. The initial amount of reagent is 10 and the initial rate is –2. Express *A* in terms of *t*.

9. The ground breaking US Civil War ironclad *Monitor* had a turret gun capable of firing a projectile up to a range of 3 650 yards (3 340 m) at an elevation of +15°. The muzzle velocity of the gun is *v* ms^{-1}.

   a Express the initial vertical and horizontal velocities of the gun in terms of *v*.

   b Find and solve the differential equation for the vertical motion of the projectile.

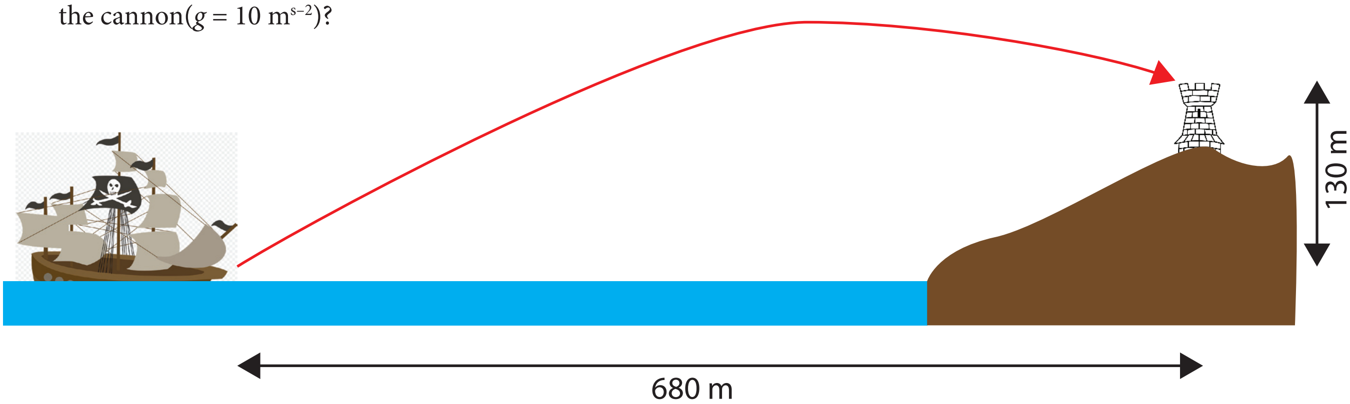
   c Find the time at which the projectile reaches its maximum height.

   d Find the time of flight of the projectile.

   e Find the muzzle velocity.

10. EXTENSION. By considering the solutions to the quadratic equation $x^2 - 3x + 2 = 0$, find the general solution of $y'' - 3y + 2 = 0$.

11. Blackbeard and his pirates are planning a raid on Castle Drake. They are offshore and want to begin with a bombardment using a cannon with a muzzle velocity of 120 ms^{-1}. At what angle(s) to the horizontal should Blackbeard's crew incline the cannon(g = 10 m^{s-2})?



Exercise A.8.3

By induction, prove that:

g $n + 1 < 3n$ for all $n \geq 1$

h $1 + n^2 < (1 + n)^2$ for all $n \geq 1$

i $n < 1 + n^2$ for all $n \geq 1$

j $7^n - 3^n$, $n \geq 1$ is divisible by 4

[Hint: $7^{k+1} - 3^{k+1} = (7^{k+1} - 7\times3^k) + (7\times3^k - 3^{k+1})$]

Exercise A.8.4

Prove the following using the principle of mathematical induction for all $n \in \mathbb{Z}^+$.

i $1.3.5 + 2.4.6 + \ldots + n(n+2)(n+4) = \frac{1}{4}n(n+1)(n+4)(n+5)$

j $\frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{(n+1)(n+2)} = \frac{n}{2(n+2)}$

k $\frac{1}{1.3} + \frac{1}{2.4} + \ldots + \frac{1}{n(n+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$

l $(1+x)^n > 1 + nx + nx^2$ for all $n \geq 3, x > 0$

m $2^n \geq n^2$ for all integers $n \geq 4$.

n $1.1! + 2.2! + 3.3! + \ldots + n.n! = (n+1)! - 1$

o $\frac{0}{1!} + \frac{1}{2!} + \frac{2}{3!} + \ldots + \frac{n-1}{n!} = 1 - \frac{1}{n!}$

p $n! > n^2$ for all $n > 3$.

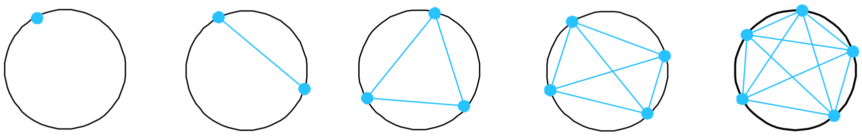
q $n^5 - n$ is divisible by 5.

r $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}} > 2(\sqrt{n+1} - 1)$, $n \geq 1$.

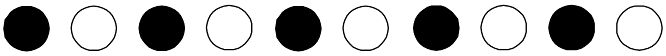
s Prove that the maximum number of points of intersection of $n \geq 2$ lines in a plane is $\frac{1}{2}n(n-1)$.

Exercise A.8.5

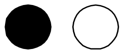
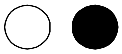
8. In each of these circles, each dot is joined to every other one. Into how many pieces is each circle divided? If a circle is drawn with n dots, what is the maximum number of regions that can be formed if all dots are joined in the same manner?



9. You are given five black discs and five white discs which are arranged in a line as shown:



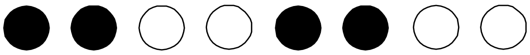
The task is to get all of the black discs to the right-hand side and all of the white discs to the left-hand side. The only move allowed is to interchange two neighbouring discs.

i.e.  becomes 

What is the smallest number of moves that need to be made?

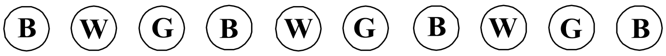
How many moves would it take if we had n black discs and n white discs arranged alternatively?

Suppose the discs are arranged in pairs.



How many moves would it take if there were n each of black and white?

Now suppose that you have three colours, black, white and green.



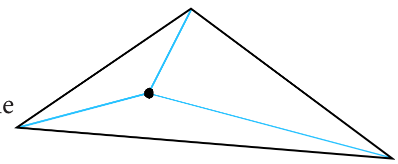
The task here is to get all the black discs to the right, all the green discs to the left and the white discs to the middle. What is the smallest number of moves required if there are n discs of each colour?

10. Prove that $\sin\theta + \sin 3\theta + \ldots + \sin(2n-1)\theta = \dfrac{\sin^2 n\theta}{\sin\theta}$.

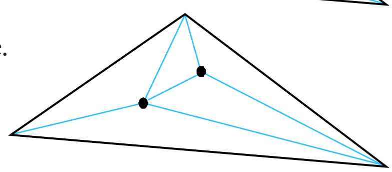
11. Prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \dfrac{(2n)!}{(n!)^2}$.

(*Hint:* consider the coefficient of x^n in the expansion of $(1+x)^n(1+x)^n$).

12. Consider placing a point inside a triangle so that non-intersecting lines are drawn from the point to the vertices of the triangle, creating partitions of the larger triangle into smaller triangles. How many partitioned triangles do we have?



Now consider the same problem as above, but this time with two points inside the larger triangle. How many partitioned triangles do we now have?



Make a proposition for this situation when n points are placed inside the triangle. Use the principle of mathematical induction to prove your proposition.

Exercise C.8.2

6. Prove $\sin^2x(1+n\cot^2x)+\cos^2x(1+n\tan^2x) = \sin^2x(n+\cot^2x)+\cos^2x(n+\tan^2x)$.

7. If $k\sec\phi = m\tan\phi$, prove that $\sec\phi\tan\phi = \dfrac{mk}{m^2-k^2}$.

8. If $x = k\sec^2\phi+m\tan^2\phi$ and $y = l\sec^2\phi+n\tan^2\phi$, prove that $\dfrac{x-k}{k+m} = \dfrac{y-l}{l+n}$.

9. Given that $\tan\theta = \dfrac{2a}{a^2-1}$, $0<\theta<\dfrac{\pi}{2}$, find: **a** $\sin\theta$ **b** $\cos\theta$

10. **a** If $\sin x+\cos x = 1$, find the values of: **i** $\sin^3x+\cos^3x$ **ii** $\sin^4x+\cos^4x$

b Hence, deduce the value of $\sin^kx+\cos^kx$, where k is a positive integer.

11. If $\tan\phi = -\dfrac{1}{\sqrt{x^2-1}}$, $\dfrac{\pi}{2}<\phi<\pi$, find, in terms of x,

a $\sin\phi+\cos\phi$ b $\sin\phi-\cos\phi$ c $\sin^4\phi-\cos^4\phi$

12. Find: **a** the maximum value of **b** the minimum value of

i $\cos^2\theta+5$ **ii** $\dfrac{5}{3\sin^2\theta+2}$ **iii** $2\cos^2\theta+\sin\theta-1$

13. **a** Given that $b\sin\phi = 1$ and $b\cos\phi = \sqrt{3}$, find b.

b Hence, find all values of ϕ that satisfy the relationship described in part **a**.

14. Find: **a** the maximum value of **b** the minimum value of

i $5^{3\sin\theta-1}$ **ii** $3^{1-2\cos\theta}$

15. Given that $\sin\theta\cos\theta = k$, find: **a** $(\sin\theta+\cos\theta)^2$, $\sin\theta+\cos\theta>0$.

b $\sin^3\theta+\cos^3\theta$, $\sin\theta+\cos\theta>0$ b

16. **a** Given that $\sin\phi = \frac{1-a}{1+a}, 0 < \phi < \frac{\pi}{2}$, find $\tan\phi$.

b Given that $\sin\phi = 1 - a, \frac{\pi}{2} < \phi < \pi$, find : **i** $2 - \cos\phi$ **ii** $\cot\phi$

17. Find:

a the value(s) of $\cos x$, where $\cot x = 4(\operatorname{cosec} x - \tan x)$, $0 < x < \pi$.

b the values of $\sin x$, where $3\cos x = 2 + \frac{1}{\cos x}, 0 \le x \le 2\pi$.

18. Given that $\sin 2x = 2\sin x\cos x$, find all values of x, such that $2\sin 2x = \tan x, 0 \le x \le \pi$.

Example 3.5.6

Give the exact value of: a $\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)+\text{Sin}^{-1}\left(\frac{3}{4}\right)\right)$ b $\cos\left(2\text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$

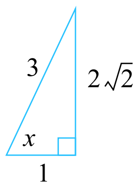
a as $\frac{1}{3} \in [-1, 1]$ and $\frac{3}{4} \in [-1, 1]$ then both $\text{Cos}^{-1}\left(\frac{1}{3}\right)$ and $\text{Sin}^{-1}\left(\frac{3}{4}\right)$ exist.

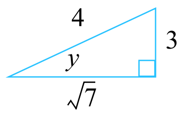
Using the compound angle formula for sine, we have that

$$\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)+\text{Sin}^{-1}\left(\frac{3}{4}\right)\right)$$

$$= \sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right)\cos\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right) + \sin\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right)\cos\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right)$$
$$= \sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right)\cos\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right) + \frac{3}{4}\times\frac{1}{3}$$

However, we now need to construct two right-angled triangles to evaluate the first part of the expression. Let $x = \text{Cos}^{-1}\left(\frac{1}{3}\right)$ and $y = \text{Sin}^{-1}\left(\frac{3}{4}\right)$ so that we have the following triangles:





Meaning that $\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)\right) = \sin x = \frac{2\sqrt{2}}{3}$

and $\cos\left(\text{Sin}^{-1}\left(\frac{3}{4}\right)\right) = \cos y = \frac{\sqrt{7}}{4}$

Therefore, $\sin\left(\text{Cos}^{-1}\left(\frac{1}{3}\right)+\text{Sin}^{-1}\left(\frac{3}{4}\right)\right) = \frac{2\sqrt{2}}{3}\times\frac{\sqrt{7}}{4}+\frac{3}{4}\times\frac{1}{3} = \frac{2\sqrt{14}+3}{12}$

As $\frac{2}{\sqrt{5}} \in [-1, 1] \Rightarrow \text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)$ exists. Next, let $\theta = \text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)$, i.e. $\text{Sin}\theta = \frac{2}{\sqrt{5}}$.

Then, using the double-angle formula for cosine, we have

$$\cos\left(2\text{Sin}^{-1}\left(\frac{2}{\sqrt{5}}\right)\right) = \cos(2\theta) = 1-2\sin^2\theta$$
$$= 1-2[\sin\theta]^2$$
$$= 1-2\left[\frac{2}{\sqrt{5}}\right]^2$$
$$= -\frac{3}{5}$$

Exercise C.8.3

6. Find the exact value of:

a $\sin\left[\text{Sin}^{-1}\left(\frac{3}{5}\right)+\text{Sin}^{-1}\left(\frac{4}{5}\right)\right]$ b $\sin\left[\text{Sin}^{-1}\left(\frac{3}{5}\right)+\text{Sin}^{-1}\left(-\frac{4}{5}\right)\right]$

c $\cos\left[\text{Tan}^{-1}\left(\frac{4}{3}\right)-\text{Cos}^{-1}\left(\frac{5}{13}\right)\right]$ d $\tan\left[\text{Tan}^{-1}\left(\frac{4}{3}\right)+\text{Tan}^{-1}\left(\frac{3}{4}\right)\right]$

e $\sin\left(2\text{Arcsin}\left(\frac{2}{3}\right)\right)$ f $\cos\left(2\text{Tan}^{-1}\left(-\frac{1}{2}\right)\right)$

g $\tan\left(2\text{Cos}^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ h $\cos\left(2\text{Sin}^{-1}\left(-\frac{1}{2}\right)\right)$

7. Prove that: a $\text{Sin}^{-1}\left(\frac{7}{25}\right) = \text{Cos}^{-1}\left(\frac{24}{25}\right)$ b $\text{Tan}^{-1}\left(\frac{1}{2}\right)+\text{Tan}^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

c $\text{Tan}^{-1}(1)+\text{Tan}^{-1}(2)+\text{Tan}^{-1}(3) = \pi$

8. Prove that: a $\text{Sin}^{-1}x = \text{Cos}^{-1}\sqrt{1-x^2},\ 0 \le x \le 1$

b $\text{Cos}^{-1}x = \text{Sin}^{-1}\sqrt{1-x^2},\ 0 \le x \le 1$

c $\text{Tan}^{-1}x+\text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x+y}{1-xy}\right)$, for all real x and y, xy ¦ 1.

d $\text{Tan}^{-1}x-\text{Tan}^{-1}y = \text{Tan}^{-1}\left(\frac{x-y}{1+xy}\right)$, for all real x and y, xy ¦ -1.

9. Find: a $\tan(\text{Cos}^{-1}k)$, where $-1 \le k \le 1,\ k \ne 0$.

b $\cos(\text{Tan}^{-1}k)$, where k is a real number.

10. State the implied domain of the following functions and sketch their graphs.

a $f(x) = \text{Sin}^{-1}\left(\frac{x}{2}\right)+\frac{\pi}{2}$ b $f(x) = \text{Cos}^{-1}(2x)-\pi$

c $g(x) = 2\text{Sin}^{-1}(x-1)$ d $h(x) = \text{Cos}^{-1}(x+2)-\frac{\pi}{2}$

11. a Prove that $\text{Sin}^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right) = \text{Tan}^{-1}x$, for all real x.

b Prove that $\text{Cos}^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right) = \text{Tan}^{-1}x$, for all real x.

12. a On the same set of axes sketch the graphs of $y = \text{Cos}^{-1}x$ and $y = \text{Sin}^{-1}x$.

b Hence, deduce the value of k, where

i $\text{Cos}^{-1}x+\text{Sin}^{-1}x = k,\ -1 \le x \le 0$

ii $\text{Cos}^{-1}x+\text{Sin}^{-1}x = k,\ 0 \le x \le 1$

On a separate set of axes, sketch the graph of $y = \text{Cos}^{-1}x+\text{Sin}^{-1}x,\ -1 \le x \le 1$.

13. Prove that if $n > 1$ then $\text{Arctan}\left(\frac{1}{n}\right) - \text{Arctan}\left(\frac{1}{n+1}\right) = \text{Arctan}\left(\frac{1}{1+n(n+1)}\right)$.

Hence, find $\sum_{i=1}^{n} \text{Arctan}\left(\frac{1}{1+i(i+1)}\right)$.

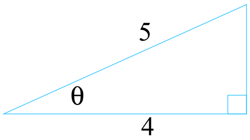
Extra Examples

Example 3.3.8

If $\sin\theta = \frac{3}{5}$ and $\cos\phi = -\frac{12}{13}$, where $0 \le \theta \le \frac{\pi}{2}$ and $\pi \le \phi \le \frac{3\pi}{2}$, find:

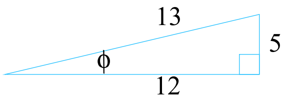
a $\sin(\theta+\phi)$ b $\cos(\theta+\phi)$ c $\tan(\theta-\phi)$

We start by drawing two right-angled triangles satisfying the given conditions:



$\cos\theta = \frac{4}{5}$

$\tan\theta = \frac{3}{4}$



$\sin\phi = \frac{5}{13}$

$\tan\phi = \frac{5}{12}$

$\sin(\theta+\phi) = \sin\theta\cos\phi + \sin\phi\cos\theta$

However, we cannot simply substitute the above ratios into this expression as we now need to consider the sign of the ratios.

As $0 \le \theta \le \frac{\pi}{2}$ then $\cos\theta = \frac{4}{5}$ and as $\pi \le \phi \le \frac{3\pi}{2}$ then $\sin\phi = -\frac{5}{13}$.

Therefore, $\sin(\theta+\phi) = \frac{3}{5} \times -\frac{12}{13} + -\frac{5}{13} \times \frac{4}{5} = -\frac{56}{65}$

$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

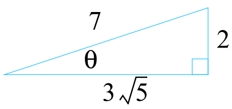
As $0 \le \theta \le \frac{\pi}{2}$ then $\cos\theta = \frac{4}{5}$ and as $\pi \le \phi \le \frac{3\pi}{2}$ then $\sin\phi = -\frac{5}{13}$.

Therefore, $\cos(\theta+\phi) = \frac{4}{5} \times -\frac{12}{13} - \frac{3}{5} \times -\frac{5}{13} = -\frac{33}{65}$

Example 3.3.9

If $\sin\theta = \frac{2}{7}$, where $\frac{\pi}{2} \le \theta \le \pi$, find: a $\sin 2\theta$ b $\cos 2\theta$ c $\tan 2\theta$

We start by drawing the relevant right-angled triangle:



a $\sin 2\theta = 2\sin\theta\cos\theta = 2 \times \frac{2}{7} \times -\frac{3\sqrt{5}}{7}$

$= -\frac{12\sqrt{5}}{49}$

b $\cos 2\theta = 1 - 2\sin^2\theta = 1 - 2 \times \left(\frac{2}{7}\right)^2 = \frac{41}{49}$

c $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta} = \frac{-\frac{12\sqrt{5}}{49}}{\frac{41}{49}} = -\frac{12\sqrt{5}}{41}$

Example 3.3.10

Prove that: a $\sin 2\alpha \tan\alpha + \cos 2\alpha = 1$ b $2\cot 2\beta = \cot\beta - \tan\beta$

a L.H.S = $\sin 2\alpha \tan\alpha + \cos 2\alpha = 2\sin\alpha\cos\alpha \times \frac{\sin\alpha}{\cos\alpha} + (1 - 2\sin^2\alpha)$

$= 2\sin^2\alpha + 1 - 2\sin^2\alpha$ = R.H.S

$= 1$

b R.H.S = $\cot\beta - \tan\beta = \frac{\cos\beta}{\sin\beta} - \frac{\sin\beta}{\cos\beta}$

$= \frac{\cos^2\beta - \sin^2\beta}{\sin\beta\cos\beta}$

$= \frac{\cos 2\beta}{\frac{1}{2}\sin 2\beta}$

$= 2\frac{\cos 2\beta}{\sin 2\beta}$

$= 2\cot 2\beta$

= L.H.S

Notice that, when proving identities, when all else fails, then express everything in terms of sine and cosine. This will always lead to the desired result – even though sometimes the working seems like it will only grow and grow – eventually, it does simplify. Be persistent.

To prove a given identity, any one of the following approaches can be used:

1. Start with the L.H.S and then show that L.H.S = R.H.S

2. Start with the R.H.S and then show that R.H.S = L.H.S

3. Show that L.H.S = p, show that R.H.S = p $\Rightarrow$ L.H.S = R.H.S

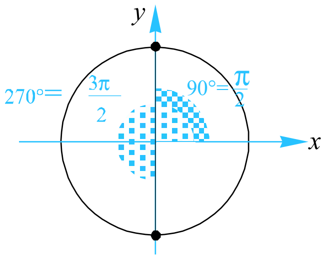
4. Start with L.H.S = R.H.S $\Rightarrow$ L.H.S – R.H.S = 0 .

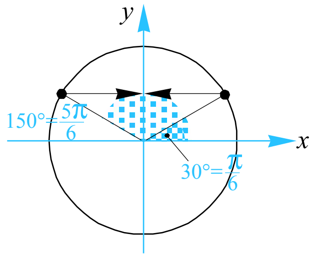
When using approaches 1 and 2, choose whichever side has more to work with.

Example 3.3.11

Find all values of x, such that $\sin 2x = \cos x$, where $0 \le x \le 2\pi$.

$\sin 2x = \cos x \Leftrightarrow 2\sin x \cos x = \cos x$





$$\Leftrightarrow 2\sin x \cos x - \cos x = 0$$
$$\Leftrightarrow \cos x(2\sin x - 1) = 0$$
$$\Leftrightarrow \cos x = 0 \text{ or } \sin x = \frac{1}{2}$$

Now, $\cos x = 0,\ 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

and

$\sin x = \frac{1}{2},\ 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$

Example 3.3.12

Simplify $\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right)$.

Express $\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where R and α are real numbers.

Hence find the maximum value of $\cos\theta - \sin\theta$.

$$\sqrt{2}\sin\left(\theta + \frac{\pi}{4}\right) = \sqrt{2}\left[\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4}\right] = \sqrt{2}\left[\sin\theta \times \frac{1}{\sqrt{2}} + \cos\theta \times \frac{1}{\sqrt{2}}\right]$$
$$= \sin\theta + \cos\theta$$

In this instance, as the statement needs to be true for all values of θ, we will determine the values of R and α by setting $R\cos(\theta + \alpha) \equiv \cos\theta - \sin\theta$.

Now, $R\cos(\theta + \alpha) = R[\cos\theta\cos\alpha - \sin\theta\sin\alpha] = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

Therefore, we have that $R\cos\theta\cos\alpha - R\sin\theta\sin\alpha \equiv \cos\theta - \sin\theta$

$$\Rightarrow R\cos\theta\cos\alpha = \cos\theta \Leftrightarrow R\cos\alpha = 1 \quad \text{– (1)}$$

$$\Rightarrow R\sin\theta\sin\alpha = \sin\theta \Leftrightarrow R\sin\alpha = 1 \quad \text{– (2)}$$

Dividing (2) by (1) we have $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{1} \Leftrightarrow \tan\alpha = 1 \therefore \alpha = \frac{\pi}{4}$

Substituting into (1) we have $R\cos\frac{\pi}{4} = 1 \Leftrightarrow R \times \frac{1}{\sqrt{2}} = 1 \therefore R = \sqrt{2}$.

Therefore, $\cos\theta - \sin\theta \equiv \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$

Then, as the maximum value of the cosine is 1, the maximum of $\sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right)$ is $\sqrt{2}$.

Exercise C.9.1

9. Prove the following identities.

p $2\csc x = \tan\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)$

q $\cos\beta + \sin\beta = \frac{\cos 2\beta}{\cos\beta - \sin\beta}$

r $\tan\alpha + \tan\beta = \frac{\sin(\alpha + \beta)}{\cos\alpha\cos\beta}$

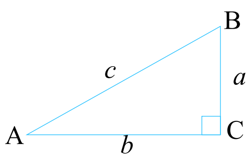
s $\sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2}$

t $\frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \frac{1}{2}\sin 2x$

10. Prove that: **a** $\frac{1 + \sin x + \cos x}{1 + \sin x - \cos x} = \cot\frac{1}{2}x$ **b** $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

c $\sin^4\phi = \frac{3}{8} + \frac{1}{8}\cos 4\phi - \frac{1}{2}\cos 2\phi$ **d** $\sin x = \frac{2\tan\frac{1}{2}x}{1 + \tan^2\frac{1}{2}x}$

11. For the right-angled triangle shown, prove that:



a $\sin 2\alpha = \frac{2ab}{c^2}$ **b** $\cos 2\alpha = \frac{b^2 - a^2}{c^2}$

c $\sin\frac{1}{2}\alpha = \sqrt{\frac{c - b}{2c}}$ **d** $\cos\frac{1}{2}\alpha = \sqrt{\frac{c + b}{2c}}$

12. Find the exact value $\tan\frac{\pi}{8}$.

13. Given that $\alpha+\beta+\gamma = \pi$, prove that $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$.

14. Solve the following for $0 \le x \le 2\pi$.

a $\sin x = \sin 2x$ **b** $\sin x = \cos 2x$ **c** $\tan 2x = 4\tan x$

15. **a** Given that $a\sin\theta + b\cos\theta \equiv R\sin(\theta+\alpha)$, express R and α in terms of *a* and *b*.

b Find the maximum value of $5 + 4\sin\theta + 3\cos\theta$.

16. **a** Given that $a\cos\theta + b\sin\theta \equiv R\cos(\theta-\alpha)$, express R and α in terms of *a* and *b*.

b Find the minimum value of $2 + 12\cos\theta + 5\sin\theta$.

17. Prove that $\tan\left(\frac{\pi}{4}+\frac{1}{2}x\right) = \sec x + \tan x$.

18. Show that if $t = \tan\frac{\pi}{12}$ then $t^2 + 2\sqrt{3}t = 1$. Hence find the exact value of $\tan\frac{\pi}{12}$.

Example

Find the angle between the lines:

$$r_1=\begin{pmatrix}2\\3\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\end{pmatrix}, r_2=\begin{pmatrix}6\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\2\end{pmatrix}$$

It is necessary to find the angle between the two vectors that represent the directions of the lines:

These are: $\begin{pmatrix}-1\\3\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}$.

Using the 'dot product method':

$$\left|\begin{pmatrix}-1\\3\end{pmatrix}\right|=\sqrt{(-1)^2+3^2}=\sqrt{10}$$

$$\left|\begin{pmatrix}1\\2\end{pmatrix}\right|=\sqrt{1^2+2^2}=\sqrt{5}$$

$$\begin{pmatrix}-1\\3\end{pmatrix}\bullet\begin{pmatrix}1\\2\end{pmatrix}=-1\times1+3\times2=5$$

$$\cos\theta=\frac{5}{\sqrt{10}\times\sqrt{5}}$$

$$\theta=45^\circ$$

Exercise C.11.6

11. The line L is defined by the parametric equations $x = 4-5k$ and $y = -2+3k$.

a Find the coordinates of three points on L.

b Find the value of k that corresponds to the point (14, –8).

c Show that the point (–1, 4) does not lie on the line L.

d Find the vector form of the line L.

e A second line, M, is defined parametrically by $x = a+10\lambda$ and $y = b-6\lambda$. Describe the relationship between M and L for the case that:

i $a = 8$ and $b = 4$ ii $a = 4$ and $b = -2$

12. Find the Cartesian equation of the line that passes through the point A(2, 1) and such that it is perpendicular to the vector $4\mathbf{i}+3\mathbf{j}$.

13. Find the direction cosines for each of the following lines:

a $\mathbf{r} = \begin{pmatrix}3\\4\end{pmatrix}+\mu\begin{pmatrix}-3\\2\end{pmatrix}$ **b** $\mathbf{r} = \begin{pmatrix}5\\9\end{pmatrix}+\lambda\begin{pmatrix}4\\3\end{pmatrix}$.

14. a Show that the line $ax + by + c = 0$ has a directional vector $\begin{pmatrix} b \\ -a \end{pmatrix}$ and a normal vector $\begin{pmatrix} a \\ b \end{pmatrix}$.

b By making use of directional vectors, which of the following lines are parallel to $L : 2x + 3y = 10$?

i $5x - 2y = 10$

ii $6x + 9y = 20$

iii $4x + 6y = -10$

15. Find the point of intersection of the lines $\boldsymbol{r} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 8 \end{pmatrix}$ and $\frac{x-3}{2} = \frac{y}{5}$.

16. Find a vector equation of the line passing through the origin that also passes through the point of intersection of the lines:

$$\boldsymbol{u} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \lambda\begin{pmatrix} -1 \\ 3 \end{pmatrix} \text{ and } \boldsymbol{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

17. Consider the line with vector equation $\boldsymbol{r} = (4\boldsymbol{i} - 3\boldsymbol{j}) + \lambda(3\boldsymbol{i} + 4\boldsymbol{j})$. Find the points of intersection of this line with the line:

a $\boldsymbol{u} = (4\boldsymbol{i} + 5\boldsymbol{j}) + \mu(2\boldsymbol{i} - \boldsymbol{j})$

b $\boldsymbol{v} = (-2\boldsymbol{i} + 3\boldsymbol{j}) + t(-6\boldsymbol{i} - 8\boldsymbol{j})$

c $\boldsymbol{w} = (13\boldsymbol{i} + 9\boldsymbol{j}) + s(3\boldsymbol{i} + 4\boldsymbol{j})$

Exercise C.11.7

8. Show that the lines $\frac{x-1}{2}=2-y=5-z$ and $\frac{4-x}{4}=\frac{3+y}{2}=\frac{5+z}{2}$ are parallel.

9. Find the Cartesian equation of the lines joining the points

a (−1, 3, 5) to (1, 4, 4) b (2, 1, 1) to (4, 1, −1)

10. a Find the coordinates of the point where the line $\mathbf{r}=\begin{pmatrix}-2\\5\\3\end{pmatrix}+t\begin{pmatrix}-1\\2\\1\end{pmatrix}$ intersects the *x*-*y* plane.

b The line $\frac{x-3}{4}=y+2=\frac{4-z}{5}$ passes through the point $(a, 1, b)$. Find the values of a and b.

11. Find the Cartesian equation of the line having the vector form:

a $\mathbf{r}=\begin{pmatrix}1\\4\\-2\end{pmatrix}+t\begin{pmatrix}1\\-1\\0\end{pmatrix}$ b $\mathbf{r}=\begin{pmatrix}2\\1\\3\end{pmatrix}+t\begin{pmatrix}2\\0\\0\end{pmatrix}$.

In each case, provide a diagram showing the lines.

12. Find the vector equation of the line represented by the Cartesian form $\frac{x-1}{2}=\frac{1-2y}{3}=z-2$.

Clearly describe this line.

13. Find the acute angle between the following lines.

a $\mathbf{r}=\begin{pmatrix}0\\2\\3\end{pmatrix}+s\begin{pmatrix}3\\4\\5\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}-2\\5\\3\end{pmatrix}+t\begin{pmatrix}-1\\2\\1\end{pmatrix}$.

b $\mathbf{r}=\begin{pmatrix}2\\1\\4\end{pmatrix}+s\begin{pmatrix}-2\\0\\1\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\1\\1\end{pmatrix}+s\begin{pmatrix}1\\1\\3\end{pmatrix}$

c $\frac{x-3}{-1}=\frac{2-y}{3}=\frac{z-4}{2}$ and $\frac{x-1}{2}=\frac{y-2}{-2}=z-2$

14. Find the point of intersection of the lines:

a $\frac{x-5}{-2} = y-10 = \frac{z-9}{12}$ and $x = 4, \frac{y-9}{-2} = \frac{z+9}{6}$

b $\frac{2x-1}{3} = \frac{y+5}{3} = \frac{z-1}{-2}$ and $\frac{2-x}{4} = \frac{y+3}{2} = \frac{4-2z}{1}$

15. Find the Cartesian form of the lines with parametric equation given by:

L : $x = \lambda, y = 2\lambda+2, z = 5\lambda$ and M : $x = 2\mu-1, y = -1+3\mu, z = 1-2\mu$

a Find the point of intersection of these two lines.

b Find the acute angle between these two lines.

c Find the coordinates of the point where: i L cuts the x-y plane. ii M cuts the y-z plane.

16. Show that the lines $\frac{x-2}{3} = \frac{y-3}{-2} = \frac{z+1}{5}$ and $\frac{x-5}{-3} = \frac{y-1}{2} = \frac{z-4}{-5}$ are coincident.

17. Show that the lines $\frac{x-1}{-3} = y-2 = \frac{7-z}{11}$ and $\frac{x-2}{3} = \frac{y+1}{8} = \frac{z-4}{-7}$ are skew.

18. Find the equation of the line passing through the origin and the point of intersection of the lines with equations $x-2 = \frac{y-1}{4}, z = 3$ and $\frac{x-6}{2} = y-10 = z-4$.

19. The lines $\frac{x}{3} = \frac{y-2}{4} = 3+z$ and $x = y = \frac{z-1}{2k}$, $k \in \mathbb{R} \setminus \{0\}$ meet at right angles. Find k.

20. Consider the lines L : $x = 0, \frac{y-3}{2} = z+1$ and M : $\frac{x}{4} = \frac{y}{3} = \frac{z-10}{-1}$.

Find, correct to the nearest degree, the angle between the lines L and M.

21. Find the value(s) of k, such that the lines $\frac{x-2}{k} = \frac{y}{2} = \frac{3-z}{3}$ and $\frac{x}{k-1} = \frac{y+2}{3} = \frac{z}{4}$ are perpendicular.

22. Find a direction vector of the line that is perpendicular to both $\frac{x+1}{3} = \frac{y+1}{8} = \frac{z+1}{12}$ and $\frac{1-2x}{-4} = \frac{3y+1}{9} = \frac{z}{6}$.

23. Are the lines $\frac{x-1}{5} = \frac{y+2}{4} = \frac{4-z}{3}$ and $\frac{x+2}{3} = \frac{y+7}{2} = \frac{2-z}{3}$ parallel? Find the point of intersection of these lines.

What do you conclude?

Exercise D.7.1

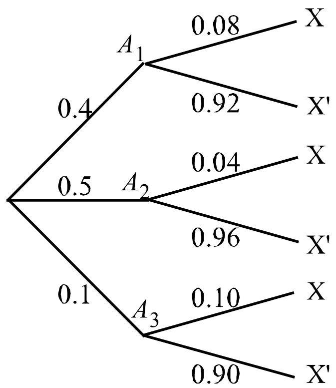
10. The probability that a patient has a virus is 0.03. A medical diagnostic test will be able to determine whether the person in question actually has the virus. If the patient has the virus, the medical test will produce a positive result 90% of the time whilst if the patient does not have the virus, it will produce a negative result 98% of the time.

    a What proportion of all tests provide a positive result?

    b If the test shows a positive result, what is the probability that the patient actually has the virus?

    c If the test shows a negative result, what is the probability that the patient does not have the virus?

11. For the tree diagram shown below, determine the probability $P(A_2|X)$.



12. Three factory employees, A, B and C, produce 40%, 30% and 30% of the total number of footballs in their division. Of these footballs, employee A produces 5% that are defective, employee B produces 6% that are defective and employee C produces 8% defectives.

    During an inspection round, a randomly selected football is found to be defective. What is the probability that employee A produced it?

13. Electrical components are checked for faults regularly at the CAMCO factory. A particular component is found to be non-defective 80% of the time, have a minor defect 12% of the time and a severe defect 8% of the time. Production levels for this component are such that 95% of the non-defective components are used for client X, 30% of the components that have a minor defect are used for client X and 5% of those that are severely defective will be used for client X.

    a Calculate the probability that a randomly selected component will be used for client X.

    b Calculate the probability that a component used for client X will have a severe defect.

14. WeCare Insurers have three types of motorcycle insurance policies, the low risk (L), moderate risk (M) and high risk (H) policies. The ratio of L to M to H policy holders is found to be 5:3:2. The respective probabilities of filing a claim by L, M and H policyholders is found to be 10%, 20% and 50% respectively.

    Calculate the probability that a policyholder who files a claim this year was a high-risk policyholder.

15. Machines M_1 , M_2 and M_3 produce 35%, 45% and 20% of the total number of bolts produced at a steel factory. It is known that each machine produces defective items. The defective items are produced by M_1 in 2% of the time, by M_2 1% of the time and by M_3 3% of the time.

    A randomly chosen item is found to be defective. Which machine is most likely to have produced it?

16. Commuters arrive at a central station on three types of trains. Sixty per cent arrive using the Express, 30% are on the Fast train and the rest arrive using the Standard train. Of those commuters arriving on the Express, half are for business-related matters. Of those arriving on the Fast train, 60% are on business-related matters and, of those coming on the Standard, 90% are on business-related matters.

Find the probability that a randomly selected commuter arriving at the station:

a is travelling for business-related matters.

b arrived using the Express train given that the person came for business-related matters.

Exercise E.7.3

1. Find the second derivative of the following functions.

m $f(x) = x^3 \sin x$ n $y = x \ln x$ o $f(x) = \frac{x^2 - 1}{2x + 3}$ p $y = x^3 e^{2x}$

q $f(x) = \frac{\cos(4x)}{e^x}$ r $y = \sin(x^2)$ s $f(x) = \frac{x}{1 - 4x^3}$ t $y = \frac{x^2 - 4}{x - 3}$

7. Find the *n*th derivative of:

a e^{ax} b $y = \frac{1}{2x + 1}$ c $\sin(ax + b)$

8. a Find $f''(2)$ if $f(x) = x^2 - \sqrt{x}$. b Find $f''(1)$ if $f(x) = x^2 \mathrm{Tan}^{-1}(x)$.

9. Find the rate of change of the gradient of the function $g(x) = \frac{x^2 - 1}{x^2 + 1}$ where $x = 1$.

10. Find the values of x where the rate of change of the gradient of the curve $y = x \sin x$ for $0 \le x \le 2\pi$ is positive.

Exercise E.7.4

1. Find the fourth derivative of:

e $f(x) = \sqrt{x+2}$

f $g(x) = \frac{x^2}{x+1}$

g $h(x) = e^{-3x} + \log(2x)$

h $g(t) = e^{2t}\cos(3t)$

i $f(x) = \frac{\log_e(x)}{x}$

7. Prove, by mathematical induction, that if:

$f(x) = \log_e(1+x)$ then,

$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}, x > -1.$

8. Guess the nth derivative, $f^{(n)}(x)$, of the function $f(x) = \sin(x)$ and prove your statement by making use of the principle of mathematical induction.

9. Prove, by mathematical induction, that if $f(x) = e^{2x}$ then $f^{(n)}(x) = 2^n e^{2x}, n \geq 1$.

10.a Guess the formula for the nth derivative of:

i $\frac{1}{1-x}$ ii $x\log(x)$

b For each case, prove your result using the principle of mathematical induction.

11. Using the principle of mathematical induction, show

that if $y = x^n \log_e(x)$ then $\frac{d^{n+1}y}{dx^{n+1}} = \frac{n!}{x}, x \neq 0$.

Exercise E.8.1

7. Determine the following limits, if they exist.

c $\lim_{x \to 0^+} (\cos x)(\ln x)$

8. Evaluate the following limits, if they exist.

(d) $\lim_{x \to 0}\left(\operatorname{cosec} x - \frac{1}{x}\right)$ (e) $\lim_{x \to 0} x^2 \ln x$ (f) $\lim_{x \to 0}\left(\frac{e^x - e^{-x} - x^2 - 2}{\sin^2 x - x^2}\right)$

(g) $\lim_{x \to 0}\left(\frac{\cot x}{\cot 2x}\right)$ (h) $\lim_{x \to \infty}\left(\frac{5x + 2\ln x}{x + 3\ln x}\right)$ (i) $\lim_{x \to 0}\left(\frac{\cos 2x - \cos x}{\sin^2 x}\right)$

9. (a) Determine i. $\lim_{x \to 8}\left(\frac{x-8}{\sqrt[3]{x} - 2}\right)$ ii. $\lim_{x \to 1}\left(\frac{e^x - e}{x - 1}\right)$

(b) Consider the continuous function f with a continuous first derivative such that $f(\pi) = 0$.

Given that $\lim_{x \to \pi} \frac{f(x)}{\sin x} = 2$, calculate the value $f'(\pi)$.

10. Determine $\lim_{x \to 0} x^{\sin x}$. [Hint: Let $z = x^{\sin x}$ and take $\ln z$ to transform it to the form $\frac{\infty}{\infty}$].

11. Show that $(1 + x)^{1/x} = e^{\left(\frac{1}{x}\right)\ln(1 + x)}$. Hence, prove that $\lim_{x \to 0^+} (1 + x)^{1/x} = e$.

12. Determine $\lim_{x \to \infty} x(a^{1/x} - 1)$.

13. Determine the following limits.

(a) $\lim_{x \to 1^+} x^{\frac{1}{x-1}}$ (b) $\lim_{x \to 0} (\sin x)^x$ (c) $\lim_{x \to \infty} (x + 1)^{2/x}$

14. Determine (a) $\lim_{x \to 0} (\cos x)^{1/x}$ (b) $\lim_{x \to 0} \left(\frac{1}{x}\right)^x$.

15. (a) Given that $f(1) = 1$ and $f'(1) = e$, calculate $\lim_{x \to 1} \frac{[f(x)]^4 - 1}{x^2 - 1}$.

(b) Given that $\lim_{x \to 0} \frac{f(x)}{e^x - 1} = e$, where f is a continuous differentiable function, find $f'(0)$ given that $f(0) = 0$.

16. Evaluate (a) $\lim_{x \to 0} \frac{\sin \alpha x}{\sin \beta x}$ (b) $\lim_{x \to 0} \frac{\sin x^\alpha}{(\sin x)^\alpha}$, where $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{Z}^+$.

17. Evaluate (a) $\lim_{x \to 0} \frac{\cos \alpha x - \cos \beta x}{x^2}$ (b) $\lim_{x \to \beta} \frac{\sin^2 x - \sin^2 \beta}{x^2 - \beta^2}$

Exercise E.9.2

17. A solid ball of radius 30 cm is dissolving uniformly in such a way that its radius is x cm, and is decreasing at a constant rate of 0.15 cm/s, t seconds after the process started.

a Find an expression for the radius of the ball at any time t seconds.

b Find the domain of x.

c Find the rate of change of:

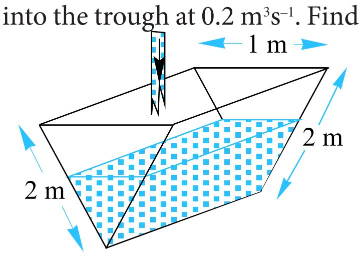
i the volume of the ball 10 seconds after it started to dissolve.

ii the surface area when the ball has a volume of 100π cm^3.

d Sketch a graph of the volume of the ball at time t seconds.

18. A fisherman is standing on a jetty and is pulling in a boat by means of a rope passing over a pulley. The pulley is 3 m above the horizontal line where the rope is tied to the boat. At what rate is the boat approaching the jetty if the rope is being hauled at 1.2 m/s, when the rope measures 12 m?

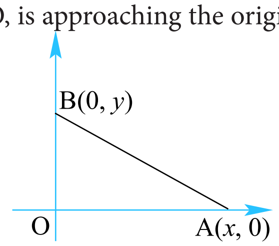
19. A trough, 4 m long, has a cross-section in the shape of an isosceles triangle. Water runs into the trough at 0.2 m^{3}s^{-1}. Find the rate at which the water level is rising after 10 seconds if the tank is initially empty.



20. A line, 12 m long, meets the x-axis at A and the y-axis at B. If point A, initially 5 m from O, is approaching the origin, O, at 2 m/s, find:

a an expression for y in terms of the time, t seconds, since point A started to move.

b the rate at which B is moving when A has travelled 2 m.



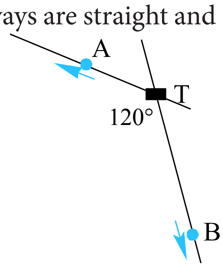
21^: The volume V cm^3 of water in a container at time t seconds, when the depth of water in the container is x cm is given by the relationship

$$V = \frac{1}{3}(x+3)^3 - 9\,,\ 0 \le x \le 5.$$

a Find the rate at which the water level is increasing after 5 seconds if water flows into the container at 1.2 cm^{3}s^{-1}.

b Find the rate of change of the area of the surface of the water after 5 seconds if water is still flowing into the container at 1.2 cm^{3}s^{-1}.

22. Two cars, A and B, leave their hometown, T, at the same time but on different freeways. The freeways are straight and at 120° to each other and the cars are travelling at 70 km/h and 80 km/hr respectively. Given that x km and y km are the distances travelled by the cars A and B respectively t hours after they leave T:



a find an expression in terms of t for the distance travelled by car:

i A ii B

b find an expression in terms of t for the distance apart cars A and B are after t hours.

c How fast are cars A and B moving apart after 5 hours?

d After travelling for 5 hours, the driver of car B decides to head back to T. How fast are the cars moving apart 3 hours after car B turns back?

23. A girl approaches a tower 75 m high at 5 km/hr. At what rate is her distance from the top of the tower changing when she is 50 m from the foot of the tower?

24. Jenny is reeling in her kite, which is maintaining a steady height of 35 m above the reel. If the kite has a horizontal speed of 0.8 m/s towards Jenny, at what rate is the string being reeled in when the kite is 20 m horizontally from Jenny?

25. A kite 60 metres high, is being carried horizontally away by a wind gust at a rate of 4 m/s. How fast is the string being let out when the string is 100 m long?

26. Grain is being released from a chute at the rate of 0.1 cubic metres per minute and is forming a heap on a level horizontal floor in the form of a circular cone that maintains a constant semi-vertical angle of 30°. Find the rate at which the level of the grain is increasing 5 minutes after the chute is opened.

27. A radar tracking station is located at ground level vertically below the path of an approaching aircraft flying at 850 km/h and maintaining a constant height of 9,000 m. At what rate in degrees is the radar rotating while tracking the plane when the horizontal distance of the plane is 4 km from the station.

28. A weather balloon is released at ground level and 2,500 m from an observer on the ground. The balloon rises straight upwards at 5 m/s. If the observer is tracking the balloon from his fixed position, find the rate at which the observer's tracking device must rotate so that it can remain in-line with the balloon when the balloon is 400 m above ground level.

29. The radius of a uniform spherical balloon is increasing at 3% per second.

   a Find the percentage rate at which its volume is increasing.

   b Find the percentage rate at which its surface area is increasing.

30. A manufacturer has agreed to produce x thousand 10-packs of high quality recordable compact discs and have them available for consumers every week with a wholesale price of $\$k$ per 10-pack. The relationship between x and k has been modelled by the equation $x^2 - 2.5kx + k^2 = 4.8$

   At what rate is the supply of the recordable compact discs changing when the price per 10-pack is set at $9.50, 4420 of the 10-pack discs are being supplied and the wholesale price per 10-pack is increasing at 12 cents per 10-pack per week?

31. It has been estimated that the number of housing starts, N millions, per year over the next 5 years will be given by:

$$N(r) = \frac{8}{1 + 0.03r^2},$$

   where r% is the mortgage rate. The government believes that over the next t months, the mortgage rate will be given by:

$$r(t) = \frac{8.6t + 65}{t + 10}.$$

   Find the rate at which the number of housing starts will be changing 2 years from when the model was proposed.

32. The volume of a right circular cone is kept constant while the radius of the base of the cone is decreasing at 2% per second. Find the percentage rate at which the height of the cone is changing.

33. The radius of a sector of fixed area is increasing at 0.5 m/s. Find the rate at which the angle in radians of the sector is changing when the ratio of the radius to the angle is 4.

Exercise E.9.3

18 A barrel is being filled with water in such a way that the volume of water, V mL, in the barrel after time t seconds is given by

$$V(t) = \frac{2}{3}\left(20t^2 - \frac{1}{6}t^3\right), 0 \le t \le 120 .$$

a Find the rate of flow into the barrel after 20 seconds.

b When will the rate of flow be greatest?

c Sketch the graph of $V(t), 0 \le t \le 120$.

19 The total cost, \$$C$, for the production of x items of a particular product is given by the linear relation $C = 600 + 20x, 0 \le x \le 100$, whilst its total revenue, \$$R$, is given by $R = x(100 - x), 0 \le x \le 100$.

a Sketch the graphs of the cost function and revenue function on the same set of axes.

b Determine the break-even points on your graph.

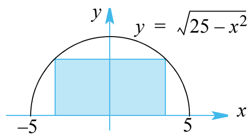
c For what values of x will the company be making a positive profit?

d Find an expression that gives the profit made in producing x items of the product and hence determine the maximum profit.

20 Find the points on the graph of $y = 9 - x^2$ that are closest to the point (0, 3).

21 A rectangle is bounded by the semicircle with equation $y = \sqrt{25 - x^2}, -5 \le x \le 5$ and the x-axis.

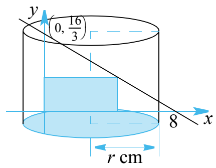
Find the dimensions of the rectangle having the largest area.



22 A rectangle is bounded by the positive x-axis the positive y-axis and the line with equation

$$y = \frac{2}{3}(8 - x)$$

Find the dimensions of the rectangle having the largest area.



23 A certificate is to be printed on a page having an area of 340 cm². The margins at the top and bottom of the page are to be 2 cm and, on the sides, 1 cm.

a If the width of the page is x cm, show that the area, A cm² where printed material is to appear is given by $A = 348 - \frac{680}{x} - 4x$.

b Hence, determine the maximum area of print.

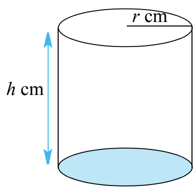
24 Find the minimum value of the sum of a positive integer and its reciprocal.

25 A closed circular cylinder is to have a surface area of 20π cm². Determine the dimensions of the cylinder which will have the largest volume.

26 A right circular cylinder of radius r cm and height h cm is to have a fixed volume of 30 cm³.

a Show that the surface area, A cm² of such a cylinder is given by $A = 2\pi r\left(r + \frac{30}{\pi r^2}\right)$.

b Determine the value of r that will yield the minimum surface area.

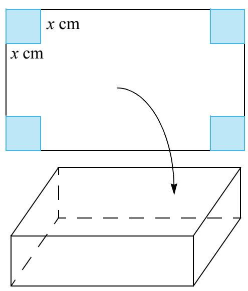


27 A rectangular container is made by cutting out squares from the corners of a 25 cm by 40 cm rectangular sheet of metal and folding the remaining sheet to form the container.

a If the squares that are cut out are x cm in length, show that the volume, V cm³ of the container is given by

$$V = x(25 - 2x)(40 - 2x), 0 < x < \tfrac{25}{2}$$

b What size squares must be cut out in order to maximize the volume of the container?



28 A right-circular cone of radius r cm contains a sphere of radius 12 cm.

a If the height of the cone is h cm, express h in terms of r.

b If V cm³ denotes the volume of the cone, find an expression for V in terms of r.

c Find the dimensions of the cone with the smallest volume.

29 For a closed cylinder of radius r cm and height h cm, find the ratio $r : h$ which will produce the smallest surface area for a fixed volume.

30 Find the coordinates of the point on the curve with equation $y = x^2 + 1$ that is closest to the point (2, 0).

31 A piece of wire is bent in the form of a sector of a circle of radius r metres containing an angle θ^c. The total length of the wire is 10 metres.

a Show that $\theta = \frac{2}{r}(5 - r)$.

b Find the value of r for which the area of this sector is a maximum.

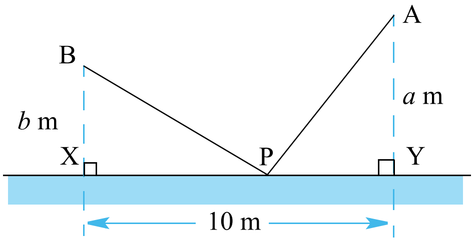
32 Find the altitude of the cylinder with the largest volume that can be inscribed in a right circular cone.

33 A window is designed so that it has an equilateral triangle mounted on a rectangular base. If the perimeter totals 7 metres, what is the maximum area, that will allow the maximum amount of light to pass through the window?

34 A cone is formed by joining the two straight edges of a sector from a circle of radius r. If the angle contained by the two straight edges is α^c find the value of α^c which makes the volume of the cone a maximum.

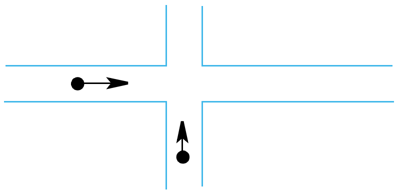
35 Two houses in a new housing estate need to have cable connected to a distribution box, P, located somewhere along a straight path XY. House A is located a m from the path while house B is located b m from the path. The cost of installing the cable is \$ $6(AP + BP)$.

Where should the box be placed so as to minimize the cost involved?



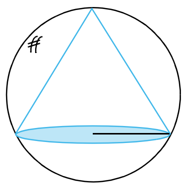
36 Felicity and Jane start walking at the same time towards an intersection of two roads that meet at right angles.

Felicity starts at 9 km from the intersection while Jane starts at 13 km from the intersection. Their speeds are 4 km/h and 3 km/h respectively. What is the closest that Felicity and Jane will get?



37 For an open cylinder of radius r cm and height h cm, find the ratio $r : h$ which will produce the smallest surface area for a fixed volume.

38 Find the height of a right circular cone which can be inscribed in a sphere of radius 1 m, if this cone is to have the largest possible volume.



39 A piece of wire 30 cm long is cut into two pieces. One of the pieces is bent into a square while the other is bent into a circle. Find the ratio of the side length of the square to the radius of the circle which provides the smallest area sum.

40 A cylindrical tin with no lid is to be made from a sheet of metal measuring 100 cm^2.

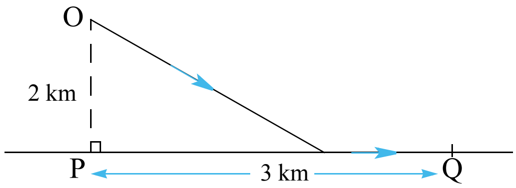
Given that the radius of the base of the tin is r cm, show that its volume, V cm^3, is given by

$$V = \frac{1}{2}(100r - \pi r^3)$$

Determine the value of r that will give the greatest volume.

41 The last leg of a triathlon requires that you get from a point O, 2 km from the nearest point P on a straight beach to a point Q, 3 km down the coast.

You may swim to any point on the beach and then run the rest of the way to point Q. If you can swim at a rate of 2 km/hr and run at 5 km/hr, where should you land on the beach so that you reach point Q in the least possible time?



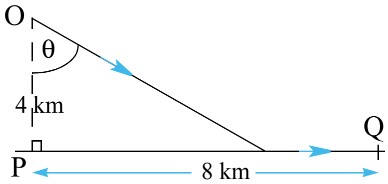
42 A right circular cylinder is inscribed in a sphere of radius 6 cm. If α is the angle subtended at the centre of the sphere by the radius of the circular base of the cylinder:

a show that the curved surface area, S cm^2, of the cylinder is given by $72\pi\sin 2\alpha$.

b find the radius and height of the cylinder having the largest curved surface area.

43 A person in a boat 4 km from the nearest point P on a straight beach, wishes to get to a point Q 8 km along the beach from P.

The person rows in a straight line to some point on the beach at a constant rate of 5 km/h. Once on the beach the person walks towards Q at a steady rate of 6 km/hr.



a Show that the total time in hours taken for the trip is

$$T(\theta) = \frac{4\sec\theta}{5} + \frac{8 - 4\tan\theta}{6},\ \theta \in \left[0, \frac{\pi}{2}\right) \text{ and } \tan\theta \le 2$$

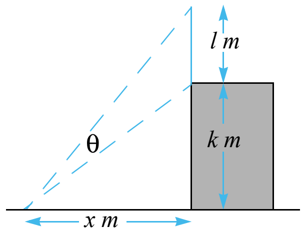
b Where should the person land so that the trip takes the least amount of time?

44 A mast l metres tall erected on a building k metres tall, subtends an angle θ at a point on the ground x metres from the base.

a Find an expression for $\tan\theta$ in terms of x.

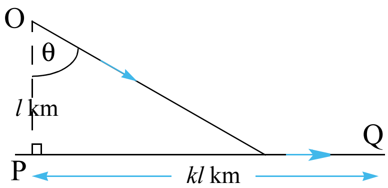
b Find the value of x that maximizes the value of θ.

(Hint: As $\theta \in \left[0, \frac{\pi}{2}\right)$, θ is a maximum when $\tan\theta$ is a maximum.)



45 A person in a boat l km from the nearest point P on a straight beach, wishes to get to a point Q kl km along the beach from P.

The person rows in a straight line to some point on the shore at a constant rate of v km/h. Once on the beach the person walks towards Q at a steady rate of u km/hr where $(v < u)$.



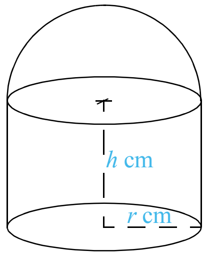
a Show that the total time in hours taken for the trip is

$$T(\theta) = \frac{l\sec\theta}{v} + \frac{kl - l\tan\theta}{u},\ \theta \in \left[0, \frac{\pi}{2}\right) \text{ and } \tan\theta \le k.$$

b Show that if the person is to reach point Q in the least possible time, then $\sin\theta = \frac{v}{u}$ where $k \ge c$, c being a particular constant.

c What would happen if $k < c$?

46 A closed tin is to be constructed as shown in the diagram. It is made up of a cylinder of height h cm and radius base r cm which is surmounted by a hemispherical cap.



a Find an expression in terms of r and h for:

i its volume, V cm^3.

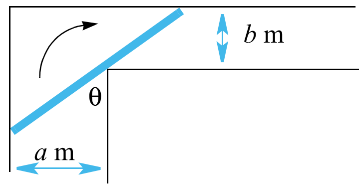
ii its surface area, A cm^2.

b Given that $V = \pi k^3, k > 0$, show that its surface area is given by $A = 2\pi k^3 \frac{1}{r} + \frac{5\pi}{3} r^2$.

c Find the ratio $r : h$ for A to be a minimum.

47 A ladder is to be carried horizontally around a corner from a corridor a m wide into a corridor b m wide.

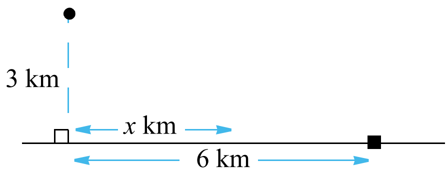
What is the maximum length that the ladder can be?



48 A man in a boat is 3 km from the nearest point of a straight beach. The man is to get to a point 6 km along the beach. The man can row at 4 km/hr and walk at 5 km/hr.

a Show that the time, T hours, taken to get to his destination is given by

$$T = \frac{\sqrt{x^2 + 9}}{4} + \frac{6 - x}{5}, \quad 0 \leq x \leq 6$$



b Where should the man row to along the beach if he is to reach his destination in the least possible time?

c After some further training, the man finds that he can now row at 4.5 km/hr. Where should he now row to along the beach to minimize the time taken?

49 The relationship between the number of spawners, S, and the number of recruits, R, in a cod farm is modelled by the equation $R = \alpha S 2^{-\beta S}$, $S \geq 0$ and the constants $\alpha > 0, \beta > 0$.

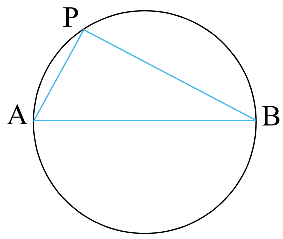
a Sketch the graphs of $R = S$ and $R = \alpha S 2^{-\beta S}$ on the same set of axes.

b For what value of S will the number of spawners first outnumber the number of recruits?

c Show that the value of S that maximizes $(R - S)$ satisfies the equation $2^{\beta S} = \alpha(1 - \beta S \ln 2)$.

50 While cruising the waters, you spot a tower 150 m tall standing on the edge of a vertical cliff 150 m high (from sea level). How far from the base of the cliff should you stand to have the tower subtend the largest possible angle at your camera lens, if the camera is in a position 5 metres above sea level?

51 The point P is joined to the ends of the diameter AB of a circle having a radius r, so that ABP forms a triangle. If P is to always remain on the circumference, what type of triangle will ABP be if:



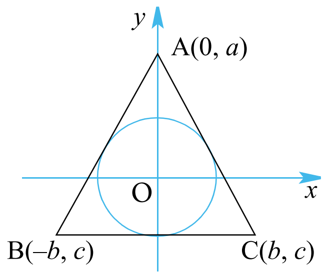
a the area of triangle ABP is a maximum?

b the perimeter of triangle ABP is a maximum?

52 An isosceles triangle ABC, where $AB = AC$ and $\angle BAC = 2\alpha$ is inscribed in a circle of radius r. Show that the area of the triangle ABC is a maximum if $\alpha = \frac{\pi}{6}$. What would the maximum area of the triangle ABC be if $0 < \alpha \leq \frac{\pi}{12}$?

53 A square PQRS of side length 1 m has three points, A, B, C on the sides QR, RS and SP such that QA = l m, RB = kl m and SC = kl m where $0 < l < 1$ and $0 < kl < 1$. If k is a constant, find the minimum value of the area of the quadrilateral PABC.

54 A variable isosceles triangle is circumscribed about a given circle of fixed radius r as shown in the diagram.



a State the value of c.

b Show that $b = \frac{r(a+r)}{\sqrt{a^2 - r^2}}$.

c Show that the area of the triangle is a minimum when the triangle is equilateral.

Exercise E.10.1

For this set of exercises, use the method of recognition to determine the integrals.

1. Find the following indefinite integrals.

g $\int \frac{4x}{\sqrt{x^2+2}}dx$ h $\int \frac{x^3}{(1-x^4)^4}dx$ i $\int 3e^{3x}\sqrt{1+e^{3x}}dx$

j $\int \frac{x+1}{(x^2+2x-1)^2}dx$ k $\int \frac{x^2+1}{\sqrt{x^3+3x+1}}dx$ l $\int x\sqrt{3+4x^2}dx$

m $\int \frac{e^x}{\sqrt{e^x+2}}dx$ n $\int \frac{e^{-2x}}{(1-e^{-2x})^3}dx$ o $\int 10x^2(x^3+1)^4 dx$

p $\int (x^3+2)(x^4+8x-3)^5 dx$ q $\int 2x^3\sqrt{(x^4+5)^3}dx$ r $\int \frac{\cos 2x}{\sqrt{1-\sin 2x}}dx$

s $\int \cos x\sqrt{4+3\sin x}dx$ t $\int \frac{\sec^2 4x}{(1+3\tan 4x)^2}dx$ u $\int \frac{1-\sin x}{\sqrt[3]{x+\cos x}}dx$

v $\int \sin\frac{1}{2}x\cos^3\frac{1}{2}x dx$ w $\int \frac{x\cos x+\sin x}{\sqrt{1+x\sin x}}dx$ x $\int \frac{(\sqrt{x}+1)^{1/2}}{\sqrt{x}}dx$

2. Find the antiderivative of the following.

g $e^x\sin(2e^x)$ h $\frac{e^{2x}}{(1-e^{2x})^2}$ i $\frac{e^{-x}}{1+e^{-x}}$

j $\frac{5}{e^{-x}+2}$ k $e^{-ax}\sqrt{4+e^{-ax}}$ l $\frac{e^{2x}}{1+e^{2x}}\ln(1+e^{2x})$

3. Find the antiderivative of the following.

g $\frac{4\sec^2 3x}{(1+\tan 3x)^2}$ h $\frac{2}{x}\cos(\ln x)$ i $\sin x\cos x\sqrt{1+\cos 2x}$

j $e^x\cos(e^x)$ k $3x^2e^{-x^3+2}$ l $\cot\frac{1}{2}x\ln\left(\sin\frac{1}{2}x\right)$

m $\sin x\sec^2 x$ n $\frac{1}{e^{-x}+2}\ln(1+2e^x)$ o $(x^2-3)\sec^2\left(\frac{1}{3}x^3-3x\right)$

4. Find the antiderivative of the following.

e $\dfrac{1}{\sqrt{16-x^2}}$

f $-\dfrac{1}{\sqrt{9-x^2}}$

5. Find the following indefinite integrals.

e $\displaystyle\int \frac{1}{\sqrt{1-4x^2}}dx$

f $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}dx$

g $\displaystyle\int \frac{1}{\sqrt{4-25x^2}}dx$

h $\displaystyle\int \frac{2}{1+4x^2}dx$

i $\displaystyle\int \frac{1}{9+4x^2}dx$

j $\displaystyle\int \frac{1}{9+16x^2}dx$

k $\displaystyle\int \frac{1}{9+5x^2}dx$

l $\displaystyle\int \frac{1}{\sqrt{3-5x^2}}dx$

6. Evaluate:

k $\displaystyle\int_{-2}^{2} \frac{x}{\sqrt{9-x^2}}dx$

l $\displaystyle\int_{1}^{2} \frac{4x}{(1+x^2)^2}dx$

m $\displaystyle\int_{0}^{1} \frac{1}{x^2+1}dx$

n $\displaystyle\int_{-1}^{1} \frac{1}{4+x^2}dx$

o $\displaystyle\int_{0}^{3} \frac{1}{1+9x^2}dx$

p $\displaystyle\int_{0}^{1/3} \frac{1}{\sqrt{1-4x^2}}dx$

q $\displaystyle\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^2}}dx$

r $\displaystyle\int_{0}^{\frac{1}{2}} \frac{1}{9x^2+1}dx$

s $\displaystyle\int_{\frac{1}{8}}^{\frac{1}{2}} \frac{1}{4+9x^2}dx$

t $\displaystyle\int_{0}^{\frac{\pi}{6}} \sin^3 x\cos x dx$

u $\displaystyle\int_{0}^{\frac{\pi}{4}} \sec x\tan^3 x dx$

v $\displaystyle\int_{1}^{e} \frac{(\ln x)^2}{x}dx$

w $\displaystyle\int_{0}^{1} \frac{2-x^2}{\sqrt{1-x^2}}dx$

x $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos 3x}{(4+\sin 3x)^2}dx$

Example

Evaluate the following:

a $\displaystyle\int_{-1}^{2} \frac{1}{x^2+2x+5}dx$ b $\displaystyle\int_{2}^{2+\sqrt{3}} \frac{1}{\sqrt{2+4x-x^2}}dx$.

Presented with the definite integral $\int_{-1}^{2} \frac{1}{x^2+2x+5}dx$, the first thing we observe is that the denominator is nearly a perfect square (sort of).

In fact, $x^2+2x+5 = (x+1)^2+4$.

So, we have $\int_{-1}^{2} \frac{1}{x^2+2x+5}dx = \int_{-1}^{2} \frac{1}{(x+1)^2+4}dx$, which is in the form suitable for an inverse tangent function.

Having recognized it as a Tan^{-1} you can make use of the substitution, $x+1 = 2\tan(\theta)$.

So, letting $x+1 = 2\tan(\theta)$ we have $\frac{dx}{d\theta} = 2\sec^2(\theta)$, when $x = -1$, $\theta = 0$ and when $x = 2$, $\theta = \frac{\pi}{4}$.

This then gives: $\displaystyle\int_{-1}^{2} \frac{1}{(x+1)^2+4}dx = \int_{0}^{\frac{\pi}{4}} \frac{1}{4\tan^2\theta+4}2\sec^2(\theta)d\theta$.

$$= \int_{0}^{\frac{\pi}{4}} \frac{2}{4(\tan^2\theta+1)}\sec^2(\theta)d\theta$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{1}{2\sec^2(\theta)}\sec^2(\theta)d\theta$$

$$= \int_{0}^{\frac{\pi}{4}} \frac{1}{2}d\theta$$

$$= \left[\frac{1}{2}\theta\right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi}{8}$$

This time we have $\displaystyle\int_{2}^{2+\sqrt{3}} \frac{1}{\sqrt{5+4x-x^2}}dx$ which is indicative of a Sin^{-1} function (after some manipulation!)

We start with some rearranging: $\int_{2}^{2+\sqrt{3}} \frac{1}{\sqrt{9-(x^2-4x+4)}}dx = \int_{2}^{2+\sqrt{3}} \frac{1}{\sqrt{9-(x-2)^2}}dx$.

This time, letting $u = x-2$, we recognize it as an inverse sine function of the form

$$\int \frac{1}{\sqrt{9-u^2}}du \text{ (with } u = x-2\text{)}.$$

So, $\int_{2}^{2+\sqrt{3}} \frac{1}{\sqrt{9-(x-2)^2}}dx = \frac{1}{3}\int_{2}^{2+\sqrt{3}} \frac{3}{\sqrt{9-(x-2)^2}}dx = \frac{1}{3}\left[\text{Sin}^{-1}\left(\frac{x-2}{3}\right)\right]_2^{2+\sqrt{3}}$

$$= \frac{1}{3}\left[\text{Sin}^{-1}\left(\frac{2+\sqrt{3}-2}{3}\right) - \text{Sin}^{-1}\left(\frac{2-2}{3}\right)\right] = \frac{1}{3}\text{Sin}^{-1}\left(\frac{\sqrt{3}}{3}\right)$$

Exercise E.10.4

1. Find the following, using the given u substitution.

h $\int t\cdot\sqrt[3]{2-t^2}\,dt$, $u = 2-t^2$

i $\int \cos x e^{\sin x}dx$, $u = \sin x$

j $\int \frac{e^x}{e^x+1}dx$, $u = e^x+1$

k $\int \cos x\sin^4 x\,dx$, $u = \sin x$

l $\int x\sqrt{x+1}\,dx$, $u = x+1$

2. Using the substitution method, find:

e $\int \frac{4x}{(1-2x^2)}dx$

f $\int \frac{4x}{(1-2x^2)^2}dx$

g $\int \frac{1}{x}(\log_e x)dx$

h $\int \frac{e^{-x}}{1+e^{-x}}dx$

i $\int \frac{1}{x\log_e x}dx$

3. Using an appropriate substitution, evaluate the following, giving exact values.

e $\int_0^{\frac{\pi}{12}} \frac{\sec^2 3x}{1+\tan 3x}dx$

f $\int_0^1 \frac{x}{(1+x^2)^2}dx$

g $\int_0^1 4x\sqrt{4+5x^2}\,dx$

h $\int_1^2 x\sqrt{x-1}\,dx$

i $\int_{-1}^1 e^x\sqrt{e^x+1}\,dx$

4. Using an appropriate substitution, evaluate the following, giving exact values.

e $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}dx$

f $\int_1^2 5xe^{(2x^2-3)}dx$

g $\int_{-1}^1 (3-2x)^7 dx$

h $\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{1}{(1-x)\sqrt{1-x^2}}dx$

i $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin 2x}dx$

5. Using an appropriate substitution, find the following, giving exact values where required.

e $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{1+\cos^2 x}dx$

f $\int_0^{\frac{\pi}{2}} \cos^3 x dx$

6. Using an appropriate substitution, find the following, giving exact values where required.

c $\int_3^6 \frac{x}{\sqrt{x-2}}dx$

d $\int_{-1}^0 \frac{x}{\sqrt{x+1}}dx$

e $\int_{-2}^0 (x-2)\sqrt{x+2}dx$

f $\int_2^5 \frac{x+1}{x-1}dx$

7. Find the following indefinite integrals.

e $\int \frac{2}{\sqrt{5+3x-x^2}}dx$

f $\int \frac{x}{\sqrt{9-x^4}}dx$

g $\int \frac{\arcsin x}{\sqrt{1-x^2}}dx$

h $\int \frac{(\arccos x)^2}{\sqrt{1-x^2}}dx$

i $\int \frac{1}{\arcsin^3 x\sqrt{1-x^2}}dx$

13. Evaluate the following definite integrals.

a $\int_0^1 \frac{2}{\sqrt{4-u^2}}du$

b $\int_{-2}^2 \frac{4}{\sqrt{9-x^2}}dx$

c $\int_0^{1/4} \frac{3}{\sqrt{1-4x^2}}dx$

d $\int_0^1 \frac{x}{\sqrt{1-x^4}}dx$

e $\int_0^1 \frac{2x-2}{\sqrt{2-x^2}}dx$

f $\int_{-1}^1 \frac{2}{4+(x+1)^2}du$

g $\int_{-3}^0 \frac{2}{x^2+6x+10}du$

Exercise E.10.5

1. Integrate the following expressions with respect to x.

e $5xe^{-4x}$ f $\ln x$ g $x\ln x$ h $-x\cos(-5x)$

i $4x\sin\left(-\frac{x}{3}\right)$ j $\frac{x}{\cos^2 x}$ k $\sqrt{x}\ln x$

2. Use integration by parts to antidifferentiate:

c $(x+1)\sqrt{x+2}$

3. Find:

c $\int \text{Sin}^{-1} x\,dx$

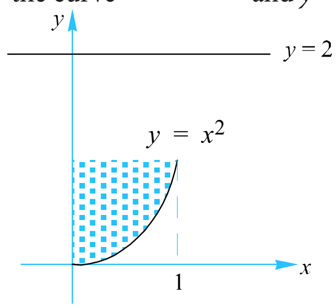
4. Find:

c $\int x\,\text{Sin}^{-1} x\,dx$

Exercise E.10.7

Extension problems

23. Find the volume of the solid of revolution generated when the shaded region shown below is revolved about the line $y=2$.



24. Find the volume of the solid of revolution generated by revolving the region enclosed by the curve $y = 4 - x^2$ and $y = 0$ about:

 a the line $y = -3$

 b the line $x = 3$

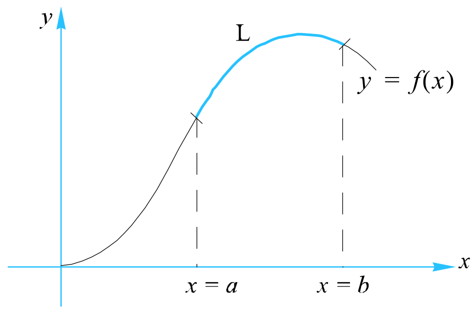
 c the line $y = 7$.

 In each case, draw the shape of the solid of revolution.

25. Show why the arc length, L units, of a curve from:

 $x = a$ to $x = b$ is given by:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx .$$



Exercise E.11.4

2. (f) $\frac{dy}{dx} + \frac{2}{x}y = \frac{\sin x}{x^2}, y(\pi) = 1, x > 0$

3. Solve the differential equation $y'\cos x = y\sin x + \sin 2x$.

4. Solve the initial-value problem $(1+x)\frac{dy}{dx} + y = 1 + x$, $y(1) = \frac{1}{4}$.

5. Solve the differential equation $\frac{dy}{dx} + y = p(x)$, where $p(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$.

6. Solve the initial-value problem $\frac{dy}{dx} + \frac{xy}{4-x^2} = 1, y(0) = 1, |x| < 2$.

7. By using the substitution $u = x^2$, show that the differential equation $\frac{dy}{dx} = \frac{x}{x^2y + y^3}$ can be transformed to the differential equation $\frac{du}{dy} - 2yu = 2y^3$.

Hence, find an implicit equation relating x and y.

8. Solve the initial-value problem $\tan x\frac{dy}{dx} + y = \sin x, y(\pi) = 0$

9. A large tank holds 100 litres of brine which is made up with 80 kg of salt. A second solution is run into the tank at a rate of 10 litres/min. The mixture is kept uniform by constantly stirring and is allowed to flow out at the same rate.
Set up and solve, the differential equation for the amount of salt present in the tank at any time t, $t \ge 0$, for each of the following cases:

(a) i. If the second solution is fresh water.
ii. If the second solution has a salt concentration of 1kg/litre.

Now consider the case where the solution is flowing out at rate of 8 litres/min.
Set up and solve, the differential equation for the amount of salt present in the tank at any time t, $t \ge 0$, for each of the following cases:

(b) i. If the second solution is fresh water.
ii. If the second solution has a salt concentration of 1kg/litre.

10. The charge Q on a charging capacitor in an RC circuit with constant voltage V is given by the differential equation

$$R\frac{dQ}{dt} + \frac{Q}{C} = V,$$

where $Q(0) = 0$. Find $Q(t)$.

11. A 10-volt battery is connected to a simple series circuit and satisfies the differential equation $\frac{1}{2}\frac{di}{dt} + 10i = 10, i(0) = 0$, where i is the current flowing in the circuit at any time t seconds.

(a) Solve the initial-value problem.
(b) Sketch the solution curve to the initial-value problem.
(c) i. Calculate the current flowing in the circuit after 0.05 sec.
ii. Determine the 'maximum' current that can flow in this circuit.

12. A differential equation used to model the height, x m, of a species of tree at age t years is given by $\frac{dx}{dt} + \alpha x = \beta(t)$ where α is a constant and $\beta(t)$ is a function of t.

For the case that $\beta(t) = \beta$ (a real constant), solve the d.e., and show that this species of tree grows to a height that will remain under $\frac{\beta}{\alpha}$ m.

Exercise E.11.17

6. Express the following functions as power series giving the interval of convergence.

(a) $f(x) = \frac{1}{2-x}$ (b) $f(x) = \frac{2}{1+4x}$ (c) $f(x) = \frac{3}{3+4x}$

7. Express the following functions as power series giving the interval of convergence.

(a) $f(x) = \frac{1}{1-x^2}$ (b) $f(x) = \frac{1}{4+x^2}$ (c) $f(x) = \ln(5-x)$

8. Express the following functions as power series giving the interval of convergence.

(a) $f(x) = \frac{x}{1-x^2}$ (b) $f(x) = \frac{x^2}{x+2}$ (c) $f(x) = \frac{x^2}{1+x^2}$

9. (a) i. Differentiate the function $g(x) = \frac{1}{2+x}$.

ii. Hence find a power series for the function $f(x) = \frac{1}{(2+x)^2}$, $-R < x < R$.

iii. Determine the value of R.

(b) i. Using part (a), find a power series for the function $h(x) = \frac{1}{(2+x)^3}$.

ii. Hence, find a power series for $q(x) = \frac{x^2}{(2+x)^3}$.

10. Evaluate the indefinite integral $\int \frac{1}{1+x^7}dx$ as a power series.

11. (a) i. Show that a power series of $\ln(3+x)$ can be expressed in the form

$$\ln a + \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{nb^n}x^n.$$

ii. Determine the value of a and b.

iii. Determine the interval of convergence for this series.

(b) Let the partial sum of this power series be $s_k(x) = \ln a + \sum_{n=1}^{k} (-1)^{n-1} \cdot \frac{1}{nb^n}x^n$.

i. Sketch, accurately, the graph of $f(x) = \ln(x+3)$.

ii. On the same set of axes, sketch the graphs of $s_2(x)$, $s_5(x)$ and $s_9(x)$.

12. Use a power series to determine

(a) $\int \frac{\arctan x}{x}dx$ (b) $\int \frac{\arctan(x^2)}{x}dx$ (c) $\int x^2\arctan(x^4)dx$

13. Find a power series for $f(x) = \ln(x+1)$ and use it to estimate $\ln(1.5)$ with an error of less than 0.0001.

14. Given that the power series for $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \ldots$

(a) determine the series for $f(x) = \sin(x^2)$.

(b) estimate $\int_0^1 \sin(x^2)dx$ to within an error of $\frac{1}{19 \times 9!}$.

15. Let the function f be defined by the power series $f(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)^3}$.

Find the intervals of convergences for (a) $f'(x)$ (b) $f''(x)$ (c) $f'''(x)$

16. Let $f(x) = e^x$,

(a) show that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all real values of x.

(b) estimate the error involved calculating e^1 if the series is evaluated using the first 10 terms.

17. (a) Determine a power series for $\frac{1}{1-x^2}$.

(b) Find a power series representing the function $f(x) = \frac{x}{(1-x^2)^2}$.

18. Determine a power series for the function $f(x) = \ln\left(\frac{1+x}{1-x}\right), |x| < 1$

19. Consider the power series $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. Determine the intervals of convergence for

(a) $f'(x)$ (b) $f''(x)$ (c) $f'''(x)$

20. (a) Show that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ for all real values of x.

(b) Hence, determine a power series for $f(x) = \cos\sqrt{x}$, $x \geq 0$.

(c) Evaluate the definite integral $\int_0^{0.1} \cos\sqrt{x}\,dx$ correct to nine decimal places.

21. Given the power series $f(x) = e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$, $g(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ and

$h(x) = \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$

(a) Show that the interval of convergence for $f(x)$, $g(x)$ and $h(x)$ are $]-\infty, \infty[$, $]-\infty, \infty[$ and $]0, 2]$ respectively.

(b) Determine a power series, including the interval of convergence for $u(x) = e^{-x}\cos x$.

(c) Using the power series, evaluate $\int_1^{1.25} (\ln x)^2 dx$ correct to 5 decimal places.

Mathematics HL Analysis and Approaches

Errata

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Solve Systems of Linear Equations with Matrices (optional)

There is another way to solve a system of linear equation using matrices. The first step is to rewrite the linear equations in standard form. Then you need to identify the coefficient matrix and the resultant matrix. If there is a solution for the system of linear equations, the solution is determined by the product between the inverse of the coefficient matrix and the resultant matrix.

Example 1.9.1

Use matrices to solve: $\begin{aligned} 3x + y - 14 &= 0 \\ -2x + 4y &= 0 \end{aligned}$

Rewrite the equations in standard form: $\begin{aligned} 3x + y &= 14 \\ -2x + 4y &= 0 \end{aligned}$

The equation can be written in matrix form as: $\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 14 \\ 0 \end{pmatrix}$

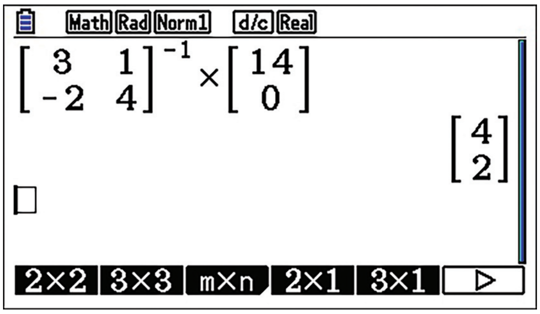
Identify the coefficient matrix: let $A = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}$

Identify the resultant matrix: let $B = \begin{pmatrix} 14 \\ 0 \end{pmatrix}$

Pre-multiplying by the inverse of A: $\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 14 \\ 0 \end{pmatrix}$

so that: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ -2 & 4 \end{pmatrix}^{-1}\begin{pmatrix} 14 \\ 0 \end{pmatrix}$

The inverse can be found using a calculator:



Therefore, the solution is $x = 4$, $y = 2$.

Mathematics HL
Analysis and Approaches

Supplementary Material

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Exercise B.5.1

1 **a** Sum = –2 Product = 4 **b** Sum = 3 Product = –7

c Sum = 4 Product = –3

d Sum = $\frac{7}{5}$ Product = $\frac{3}{5}$ **e** Sum = $\frac{-5}{2}$ Product = $\frac{-3}{2}$

f Sum = $\frac{4}{9}$ Product = $\frac{-2}{9}$ **g** Sum = $\frac{7}{3}$ Product = $\frac{4}{3}$

h Sum = $\frac{-8}{3}$ Product = $\frac{-13}{5}$ **i** Sum = $\frac{3}{4}$ Product = $\frac{-1}{8}$

Consider the possibility of a zero denominator!

2 **a** 3 **b** –1 **c** $\frac{1}{2}$ **d** $\frac{-1}{2}$ **e** $\frac{-5}{3}$

f $\frac{9}{5}$ **g** $\frac{3}{2}$ **h** 4 **i** 7

3 The cubic case: $ax^3+bx^2+cx+d=0$ gives $x^3+\frac{b}{a}x^2+\frac{c}{a}x+\frac{d}{a}=0$.

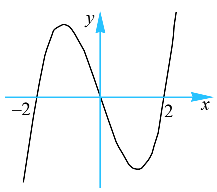
The factorized version is: $(x-\alpha)(x-\beta)(x-\gamma)=0$.

The only simple conclusion is that the product of the roots is $\alpha\beta\gamma=-\frac{d}{a}$.

4 This is related to the conjugate root theorem. The coefficients must be real.

Exercise B.5.2

1 **a**



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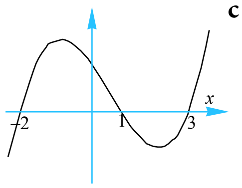

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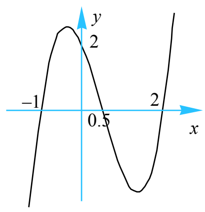

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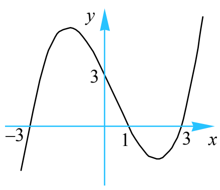

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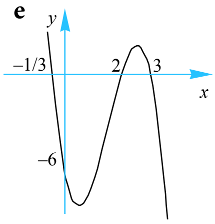

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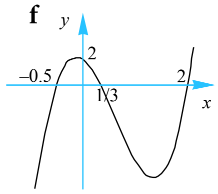

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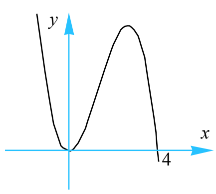

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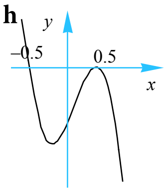

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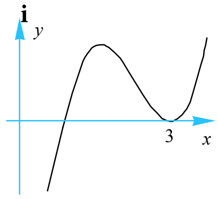

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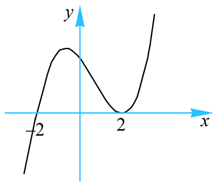

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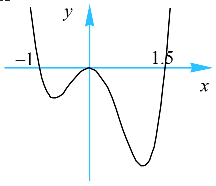

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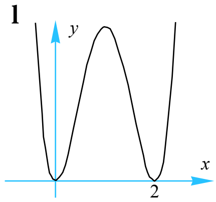

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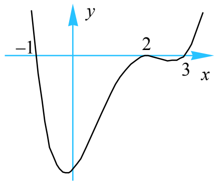

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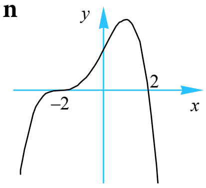

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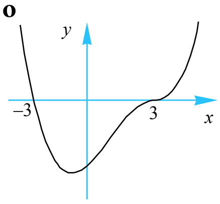

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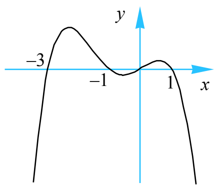

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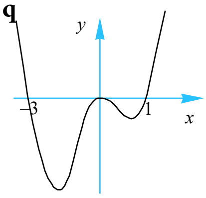

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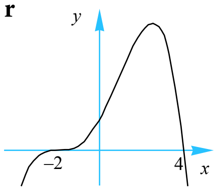

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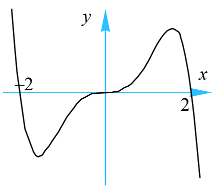

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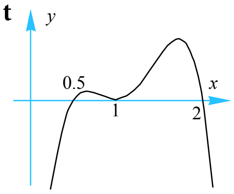
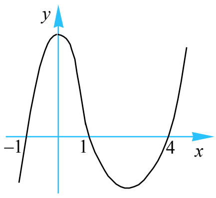


2 **a**



b

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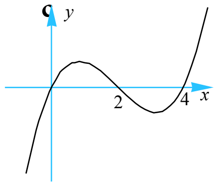

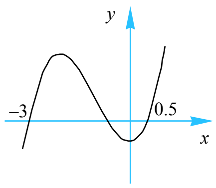

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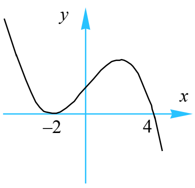

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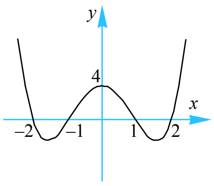

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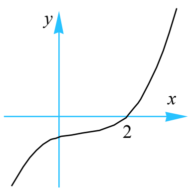

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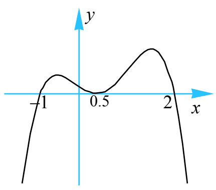

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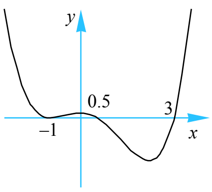

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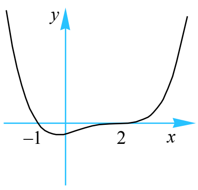

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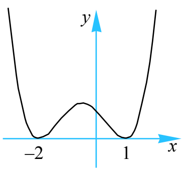
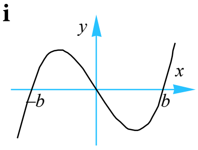
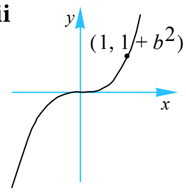


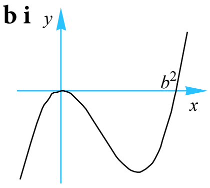
3 **a i**



ii



b i



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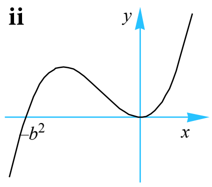


4

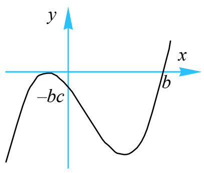
a $y = -\frac{1}{15}(x+3)(x-1)(x-5)$ **b** $y = \frac{1}{8}(x-2)^2(x+4)$ **c** $y = -\frac{3}{2}x^2(x-3)$

d $y = \frac{1}{30}(x+2)^2(75-29x)$ **e** $y = \frac{1}{6}(x+2)(4-3x)(x-3)$

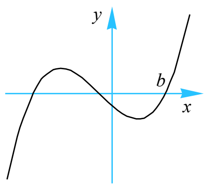
f $y = -x^3 - x^2 + 2x + 8$

5 **a** $y = \frac{1}{2}(x+2)(x-2)^3$ **b** $y = \frac{1}{35}x^2(x-3)(x-5)$ **c** $y = -\frac{1}{6}(x+2)^2(x-1)(x-3)$

6 **a**



b



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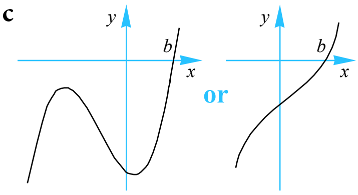
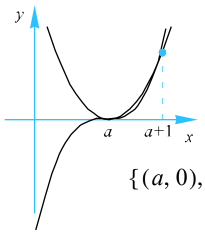


7 **a**



b $\{x : x > a + 1\}$

Exercise B.7.1

1	a	even	b	even	c	neither	d	neither
	e	even	f	odd	g	odd	h	even
	i	odd						

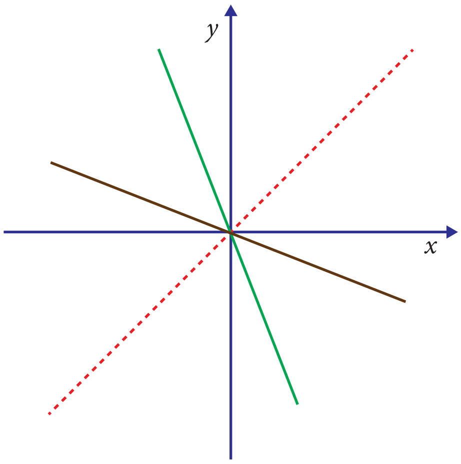
3 Not if 0 is excluded from the domain.

6 $f(x) = 0, x \in \Re$

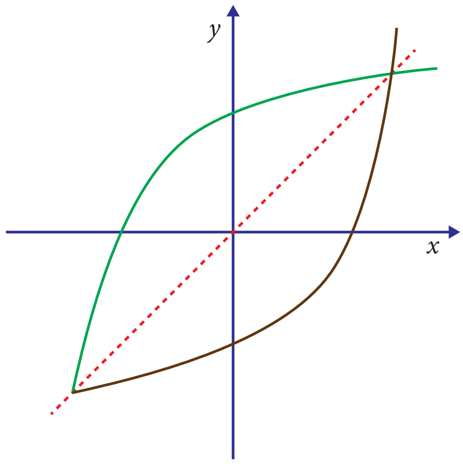
Exercise B.7.2

1. Inverse in brown.

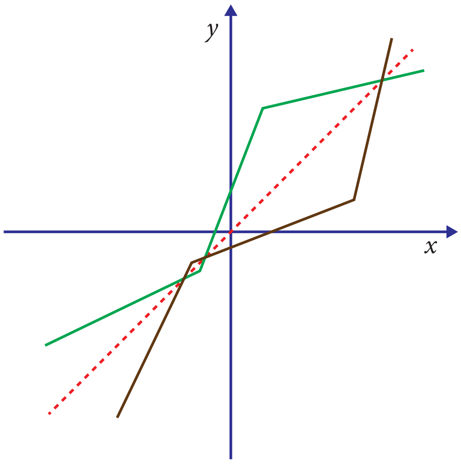
a



b

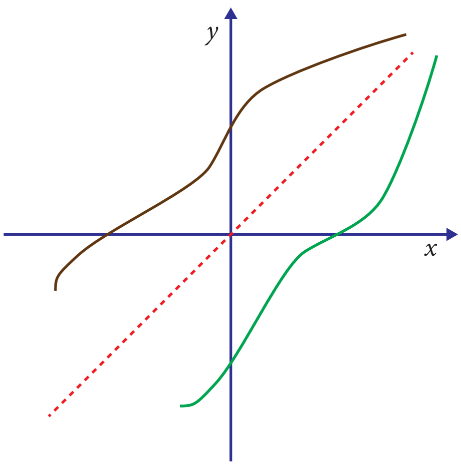


c

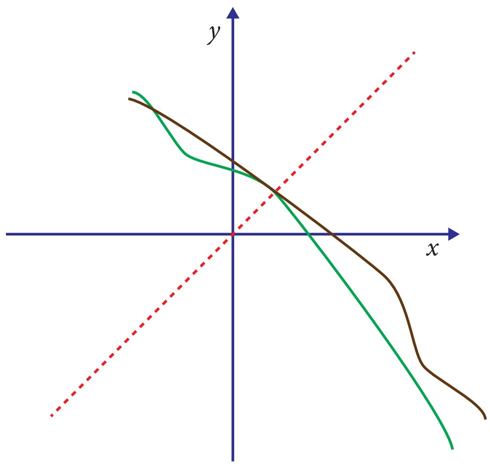


2. Answers not unique.

a



b



3. $f^{-1}(x) = x^2 + 1, x \geq 0$

4. 2

5. $f^{-1}(x)=\frac{\ln(x+1)}{2}, x\geq -1$

6. $f^{-1}(x)=\sqrt{\log_2(x+3)}, x\geq -2$

7. 1

8. 2

Exercise B.7.3

1.	a	1	b	0.785	c	no solutions
	d	1.0926	e	4.193	f	−1.710
	g	2.379	h	3.314	i	0.2146
	j	0.6417	k	−1.780	l	−0.9813, 4.749

2. a $C_A = 340\times 0.9^t, t\geq 0, C_A = 25t, t\geq 0$ b 6.708 hrs or 6 hrs 42 min.

3. a $A=10000\times 1.005^n$ n = number of months. b 44.7 months

4. a = 3 is trivial, a ≈2.478.

5. a $A_{\text{green}}=\frac{1}{2}\sin\theta, 0\leq\theta\leq 2\pi$ b $A_{\text{red \& green}}=\frac{\theta}{2}, 0\leq\theta\leq 2\pi$

   c $A_{\text{red}}=\frac{\theta}{2}-\frac{1}{2}\sin\theta, 0\leq\theta\leq 2\pi$ d $\frac{\theta}{2}-\frac{1}{2}\sin\theta=\frac{\pi}{4}$

   e 1.627 rad.

6. a $A(x)=2.4\times\ln(3.7x)$ b $B(x)=1.6^x$ c 0.453 or 3.965 - both outside the data range.

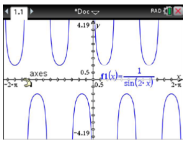
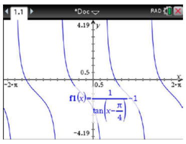
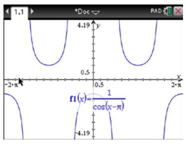
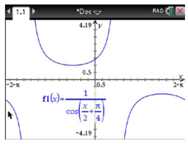
7. 19.1 months

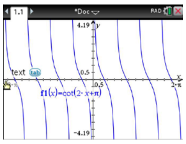
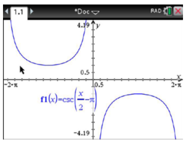
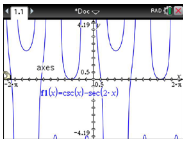
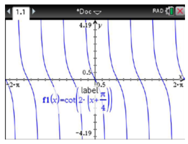
8. 1.3

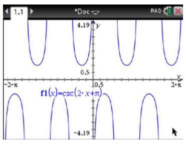
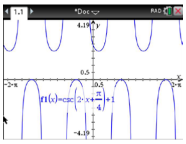
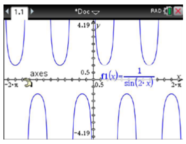
Exercise 3.2.1

1	**a**	120°	**b**	108°	**c**	216°	**d**	50°
2	**a**	π^c	**b**	$\frac{3\pi^c}{2}$	**c**	$\frac{7\pi^c}{9}$	**d**	$\frac{16\pi^c}{9}$
3	**a**	$\frac{\sqrt{3}}{2}$	**b**	$-\frac{1}{2}$	**c**	$-\sqrt{3}$	**d**	-2
	e	$-\frac{1}{2}$	**f**	$-\frac{\sqrt{3}}{2}$	**g**	$\frac{1}{\sqrt{3}}$	**h**	$\sqrt{3}$
	i	$-\frac{1}{\sqrt{2}}$	**j**	$-\frac{1}{\sqrt{2}}$	**k**	1	**l**	$-\sqrt{2}$
	m	$-\frac{1}{\sqrt{2}}$	**n**	$\frac{1}{\sqrt{2}}$	**o**	-1	**p**	$\sqrt{2}$
	q	0	**r**	1	**s**	0	**t**	undefined
4	**a**	0	**b**	-1	**c**	0	**d**	-1
	e	$\frac{1}{\sqrt{2}}$	**f**	$-\frac{1}{\sqrt{2}}$	**g**	-1	**h**	$\sqrt{2}$
	i	$-\frac{1}{2}$	**j**	$-\frac{\sqrt{3}}{2}$	**k**	$\frac{1}{\sqrt{3}}$	**l**	$\sqrt{3}$
	m	$-\frac{\sqrt{3}}{2}$	**n**	$\frac{1}{2}$				
5	**a**	$\frac{1}{2}$	**b**	$\frac{\sqrt{3}}{2}$	**c**	11	**d**	$\frac{1}{2}$
	e	$-\frac{1}{\sqrt{3}}$	**f**	$-\frac{1}{2}$	**g**	$-\sqrt{2}$	**h**	$-\frac{2}{\sqrt{3}}$
6	**a**	$-\frac{1}{2}$	**b**	$-\frac{1}{\sqrt{2}}$	**c**	$\sqrt{3}$	**d**	-2
	e	1	**f**	$\frac{1}{2}$	**g**	$-\frac{1}{\sqrt{3}}$	**h**	$-\frac{\sqrt{3}}{2}$

i $-\frac{2}{\sqrt{3}}$ j $\frac{1}{\sqrt{3}}$ k $\frac{2}{\sqrt{3}}$ l $-\frac{\sqrt{3}}{2}$

7. a  b  c  d 

e  f  g  h 

i  j  k 

Exercise C.8.3

1 a $\frac{\pi}{4}$ b $\frac{\pi}{2}$ c π d $\frac{\pi}{3}$

e $\frac{\pi}{4}$ f $-\frac{\pi}{3}$ g 1.1071^c h -0.7754^c

i 0.0997^c j 1.2661^c k -0.6435^c l 1.3734^c

m undefined n -1.5375^c o 1.0654^c

2 a -1 b $\frac{\sqrt{3}}{4}$ c $-\frac{1}{3\sqrt{2}}$

4 $\frac{1}{3}, \frac{1}{2}$

5 a $\frac{2}{3}$ b $\frac{1}{3}$ c $\frac{1}{2}$

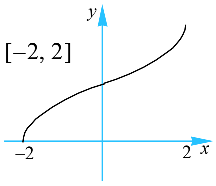
d $\frac{3}{4}$ e $\frac{3\sqrt{2}}{4}$ f -1

6 a 1 b $-\frac{7}{25}$ c $\frac{63}{65}$ d undefined

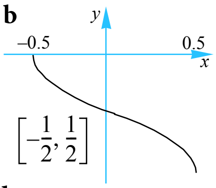
e $\frac{4\sqrt{5}}{9}$ f $\frac{3}{5}$ g $\frac{4}{3}$ h $\frac{1}{2}$

9 a $\frac{\sqrt{1-k^2}}{k}$ b $\frac{1}{\sqrt{1+k^2}}$

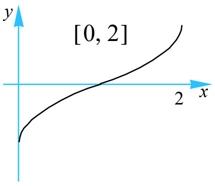
10 a



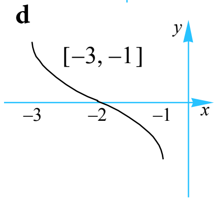
b



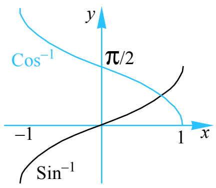
c



d

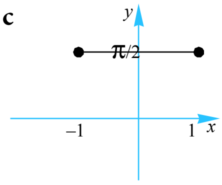


12 a



b i $\frac{\pi}{2}$ **ii** $\frac{\pi}{2}$

c



13 $\frac{\pi}{4} - \tan^{-1}\left(\frac{1}{n+1}\right)$

Exercise C.9.1

1

a $\sin\alpha\cos\phi + \cos\alpha\sin\phi$ **b** $\cos3\alpha\cos2\beta - \sin3\alpha\sin2\beta$ **c** $\sin2x\cos y - \cos2x\sin y$

d $\cos\phi\cos2\alpha + \sin\phi\sin2\alpha$ **e** $\dfrac{\tan2\theta - \tan\alpha}{1 + \tan2\theta\tan\alpha}$ **f** $\dfrac{\tan\phi - \tan3\omega}{1 + \tan\phi\tan3\omega}$

2

a $\sin(2\alpha - 3\beta)$ **b** $\cos(2\alpha + 5\beta)$ **c** $\sin(x + 2y)$

d $\cos(x - 3y)$ **e** $\tan(2\alpha - \beta)$ **f** $\tan x$

g $\tan\left(\frac{\pi}{4} - \phi\right)$ **h** $\sin\left(\frac{\pi}{4} + \alpha + \beta\right)$ **i** $\sin2x$

3 **a** $-\frac{56}{65}$ **b** $\frac{33}{65}$ **c** $-\frac{16}{63}$

4 **a** $\frac{16}{65}$ **b** $\frac{63}{65}$ **c** $\frac{56}{33}$

5 **a** $-\frac{5\sqrt{11}}{18}$ **b** $-\frac{7}{18}$ **c** $\frac{5\sqrt{11}}{7}$ **d** $\frac{35\sqrt{11}}{162}$

6 **a** $-\frac{3}{5}$ **b** $-\frac{4}{5}$ **c** $\frac{3}{4}$ **d** $\frac{24}{7}$

7 **a** $\frac{1+\sqrt{3}}{2\sqrt{2}}$ **b** $\frac{1+\sqrt{3}}{2\sqrt{2}}$ **c** $-\frac{1+\sqrt{3}}{2\sqrt{2}}$ **d** $\sqrt{3} - 2$

8 **a** $\frac{2ab}{a^2+b^2}$ **b** $\frac{a^2+b^2}{2ab}$ **c** $\frac{a^4 - 6a^2b^2 + b^4}{(a^2+b^2)^2}$ **d** $\frac{2ab}{b^2-a^2}$

12 $\sqrt{2} - 1$

14 **a** $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}, 2\pi$ **b** $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ **c** $0, \pi, 2\pi, \alpha, \pi \pm \alpha, 2\pi - \alpha, \alpha = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$

15 **a** $R = \sqrt{a^2+b^2}, \tan\alpha = \frac{b}{a}$ **b** 10

16 **a** $R = \sqrt{a^2+b^2}, \tan\alpha = \frac{b}{a}$ **b** -11

18 $2 - \sqrt{3}$

Exercise E.10.1

1 **a** $-\frac{3}{x^4}$ **b** $\frac{3}{2}\sqrt{x}$ **c** $\frac{5}{2}\sqrt{x^3}$ **d** $\frac{1}{3\sqrt[3]{x^2}}$

e $\frac{2}{\sqrt{x}}$ **f** $9\sqrt{x}$ **g** $\frac{1}{\sqrt{x}}+\frac{3}{x^2}$ **h** $\frac{3}{2}\sqrt{x}-\frac{1}{2\sqrt{x^3}}$

i $\frac{10}{3\sqrt[3]{x}}-9$ **j** $5-\frac{1}{2\sqrt{x}}-\frac{8}{5x^3}$ **k** $\frac{4}{\sqrt{x}}-\frac{15}{x^6}+\frac{1}{2}$ **l** $-\frac{1}{2\sqrt{x^3}}-\frac{1}{\sqrt{x}}+x^2$

2 **a** $\frac{3}{2}\sqrt{x}+\frac{1}{\sqrt{x}}$ **b** $4x^3+3x^2-1$ **c** $3x^2+1$ **d** $\frac{1}{x^2}$

e $\frac{1}{\sqrt{x^3}}$ **f** $\frac{1}{2}-\frac{1}{4\sqrt{x^3}}$ **g** -7 **h** $2x-\frac{8}{x^3}$

i $2x-\frac{2}{x^2}-\frac{4}{x^5}$ **j** $\frac{1}{2}\sqrt{\frac{3}{x}}+\frac{1}{6\sqrt{x^3}}$ **k** $2x-\frac{12}{5}\sqrt[5]{x}+\frac{2}{5\sqrt[5]{x^3}}$

l $-\frac{3}{2\sqrt{x}}\left(\frac{1}{x}+1\right)\left(\frac{1}{\sqrt{x}}-\sqrt{x}\right)^2$

Exercise E.10.2

1 **a** $48t^3-\frac{1}{2\sqrt{t}}$ **b** $2n-\frac{2}{n^2}-\frac{4}{n^5}$ **c** $\frac{3}{2}\sqrt{r}+\frac{5}{6\sqrt[6]{r}}-\frac{1}{\sqrt{r}}$

d $2\theta-\frac{9}{2}\sqrt{\theta}+3-\frac{1}{2\sqrt{\theta}}$ **e** $40-3L^2$ **f** $-\frac{100}{v^3}-1$

g $6l^2+5$ **h** $2\pi+8h$

2 **a** $\frac{8}{3t^3}$ **b** $2\pi r-\frac{20}{r^2}$ **c** $\frac{5}{2}s^{3/2}+\frac{3}{s^2}$

d $-\frac{6}{t^4}+\frac{2}{t^3}-\frac{1}{t^2}$ **e** $-\frac{4}{b^2}+\frac{1}{2b^{3/2}}$ **f** $3m^2-4m-4$

Exercise E10.3

1 **a** $3x^2-5x^4+2x+2$ **b** $6x^5+10x^4+4x^3-3x^2-2x$

c $-\frac{4}{x^5}$ **d** $6x^5+8x^3+2x$

2 **a** $-\frac{2}{(x-1)^2}$ **b** $\frac{1}{(x+1)^2}$ **c** $\frac{1-x^2-2x}{(x^2+1)^2}$

d $\frac{-(x^4+3x^2+2x)}{(x^3-1)^2}$ **e** $\frac{2x^2+2x}{(2x+1)^2}$ **f** $\frac{1}{(1-2x)^2}$

3 **a** $(\sin x+\cos x)e^x$ **b** $\ln x+1$ **c** $e^x(2x^3+6x^2+4x+4)$

d $4x^3\cos x - x^4\sin x$ **e** $-\sin^2 x+\cos^2 x$ **f** $2x\tan x+(1+x^2)\sec^2 x$

g $\frac{4}{x^3}(x\cos x-2\sin x)$ **h** $e^x(x\cos x+x\sin x+\sin x)$

i $(\ln x+1+x\ln x)e^x$

4 **a** $\frac{\sin x - x\cos x}{\sin^2 x}$ **b** $\frac{-[\sin x(x+1)+\cos x]}{(x+1)^2}$ **c** $\frac{e^x}{(e^x+1)^2}$

d $\frac{2x\cos x-\sin x}{2x\sqrt{x}}$ **e** $\frac{\ln x-1}{(\ln x)^2}$ **f** $\frac{(x+1)-x\ln x}{x(x+1)^2}$

g $\frac{xe^x+1}{(x+1)^2}$ **h** $\frac{-2}{(\sin x-\cos x)^2}$ **i** $\frac{x^2-x+2x\ln x}{(x+\ln x)^2}$

5 **a** $-5e^{-5x}+1$ **b** $4\cos 4x+3\sin 6x$ **c** $-\frac{1}{3}e^{-\frac{1}{3}x}-\frac{1}{x}+18x$

d $25\cos 5x+6e^{2x}$ **e** $4\sec^2 4x+2e^{2x}$ **f** $-4\sin(4x)+3e^{-3x}$

g $\frac{4}{4x+1}-1$ **h** 0 **i** $\frac{1}{2}\cos\left(\frac{x}{2}\right)-2\sin 2x$

j $7\cos(7x-2)$ **k** $\frac{1}{2\sqrt{x}}-\frac{1}{x}$ **l** $\frac{1}{x}+6\sin 6x$

6 **a** $2x\cos x^2+2\sin x\cos x$ **b** $2\sec^2 2\theta-\frac{\cos\theta}{\sin^2\theta}$ **c** $\frac{1}{2\sqrt{x}}\cos\sqrt{x}$

d $\frac{1}{x^2}\sin\left(\frac{1}{x}\right)$ **e** $-3\sin\theta\cdot\cos^2\theta$ **f** $e^x\cos(e^x)$

g $\frac{1}{x}\sec^2(\log_e x)$ **h** $\frac{-\sin 2x}{\sqrt{\cos 2x}}$ **i** $-\cos\theta\cdot\sin(\sin\theta)$

j $4\sin\theta\cdot\sec^2\theta$ **k** $-5\cos 5x\cdot\csc^2(5x)$ **l** $-6\csc^2(2x)$

7 **a** $2e^{2x+1}$ **b** $-6e^{4-3x}$ **c** $-12xe^{4-3x^2}$

d $\frac{1}{2}\sqrt{e^x}$ **e** $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ **f** e^{2x+4}

g $2xe^{2x^2+4}$ **h** $-\frac{6}{e^{3x+1}}$ **i** $(6x-6)e^{3x^2-6x+1}$

j $\cos(\theta)e^{\sin\theta}$ **k** $2\sin(2\theta)e^{-\cos 2\theta}$ **l** $2x$

m $\frac{2e^{-x}}{(e^{-x}+1)^2}$ **n** $3(e^x+e^{-x})(e^x-e^{-x})^2$ **o** e^{x+2}

p $(-2x+9)e^{-x^2+9x-2}$

8 **a** $\frac{2x}{x^2+1}$ **b** $\frac{\cos\theta+1}{\sin\theta+\theta}$ **c** $\frac{e^x+e^{-x}}{e^x-e^{-x}}$ **d** $-\frac{1}{x+1}$

e $\frac{3}{x}(\ln x)^2$ **f** $\frac{1}{2x\sqrt{\ln x}}$ **g** $\frac{1}{2(x-1)}$ **h** $\frac{-3x^2}{1-x^3}$

i $-\frac{1}{2(x+2)}$ **j** $\frac{-2\sin x\cos x}{\cos^2 x+1}$ **k** $\frac{1}{x}+\cot x$ **l** $\frac{1}{x}+\tan x$

9 **a** $\ln(x^3+2)+\frac{3x^3}{x^3+2}$ **b** $\frac{\sin^2 x}{2\sqrt{x}}+2\sqrt{x}\sin x\cos x$ **c** $-\frac{1}{\sqrt{\theta}}\sin\sqrt{\theta}\cdot\cos\sqrt{\theta}$

d $(3x^2-4x^4)e^{-2x^2+3}$ **e** $-(\ln x+1)\sin(x\ln x)$ **f** $\frac{1}{x\ln x}$

g $\frac{(2x-4)\cdot\sin(x^2)-2x\cdot\cos(x^2)(x^2-4x)}{(\sin x^2)^2}$ **h** $\frac{10(\ln(10x+1)-1)}{[\ln(10x+1)]^2}$

i $(\cos 2x-2\sin 2x)e^{x-1}$ **j** $2x\ln(\sin 4x)+4x^2\cot 4x$ **k** $(\cos\sqrt{x}-\sin\sqrt{x})\frac{1}{2\sqrt{x}}e^{-\sqrt{x}}$

l $-(2\sin x+2x\cos x)\cdot\sin(2x\sin x)$ **m** $\frac{e^{5x+2}(9-20x)}{(1-4x)^2}$ **n** $\frac{\cos^2\theta+\sin^2\theta\ln(\sin\theta)}{\sin\theta\cos^2\theta}$

o $\frac{x+2}{2(x+1)\sqrt{x+1}}$ **p** $\frac{2x^2+2}{\sqrt{x^2+2}}$ **q** $\frac{10x^3+9x^2+4x+3}{3(x+1)^{2/3}}$

r $\frac{3x^2(3x^3+1)}{2\sqrt{x^3+1}}$ **s** $\frac{2}{x^2+1}-\frac{1}{x^2}\ln(x^2+1)$ **t** $\frac{2}{x(x+2)}$

u $\dfrac{2-x}{2x^2\sqrt{x-1}}$ **v** $\dfrac{-x^2+x-9}{\sqrt{x^2+9}}\cdot e^{-x}$ **w** $\dfrac{7x^3-12x^2-8}{2\sqrt{2-x}}$

x $nx^{n-1}\ln(x^n-1)+\dfrac{nx^{2n-1}}{x^n-1}$

10 $x=1$

11 0

12 0

13 1

14 $-2e$

15 **a** $\cos^2x-\sin^2x$ **b** $\dfrac{\pi}{180}\cos x°$ **c** $-\dfrac{\pi}{180}\sin x°$

16 **b i** $2x\sin x\cos x+x^2\cos^2x-x^2\sin^2x$ **ii** $e^{-x^3}(2\cos2x\ln\cos x-3x^2\sin2x\ln\cos x-\sin2x\tan x)$

17 **a i** $-\dfrac{3}{x}(\ln x)^2$ **ii** $-\dfrac{3x^2}{1-x^3}$ **b i** $-2e^{-2x}\cdot\cos(e^{-2x})$ **ii** $-2x\cos x^2\cdot e^{-\sin x^2}$

18 $-\dfrac{1}{5}k$

19 $x=a, b, \dfrac{mb+na}{m+n}$

20 $\{\theta : n\tan\theta^m\cdot\tan\theta^n=m\theta^{m-n}\}$

21 **a** $-4\csc(4x)$ **b** $2\sec(2x)\tan(2x)$ **c** $3\cot(3x)\csc(3x)$ **d** $-3\sin(3x)$

e $\csc^2\left(\dfrac{\pi}{4}-x\right)$ **f** $-2\sec(2x)\tan(2x)$

22 **a** $2x\sec(x^2)\tan(x^2)$ **b** $\sec^2x$ **c** $\tan x$

d $-3\cot^2x\csc^2x$ **e** $x\cos x+\sin x$ **f** $-2\cot x\csc^2x$

g $4x^3\csc(4x)-4x^4\cot(4x)\csc(4x)$ **h** $2\cot x\sec^2(2x)-\csc^2x\tan(2x)$

i $\dfrac{\sec x\tan x-\sin x}{2\sqrt{\cos x+\sec x}}$

23 **a** $e^{\sec x}\sec x\tan x$ **b** $e^x\sec(e^x)\tan(e^x)$ **c** $e^x\sec(x)+e^x\sec(x)\tan(x)$

d $\dfrac{-\csc^2(\log x)}{x}$ **e** $-5\csc(5x)\sec(5x)$ **f** $\dfrac{\cot(x)}{x}-\csc^2(x)\log x$

g $-\cos x\cot(\sin x)\csc(\sin x)$ **h** $-\cos(\csc x)\cot x\csc x$ **i** 0

Exercise E.10.4

1 **a** $4\pi r$ cm^{2}s^{-1} **b** 4π cms^{-1}

2 6 cm^{2}s^{-1}

3 **a** $\frac{dA}{dt} = -\frac{3}{2}\sqrt{2}x$ cm^{2}s^{-1} (x = side length) **b** $-\frac{3}{2}\sqrt{2}$ cms^{-1}

4 **a** 37.5 cm^{3}h^{-1} **b** 30 cm^{2}h^{-1} **c** 0.96 g^{-1}cm^{3}h^{-1}

5 ~ 0.37 cms^{-1}

6 –0.24 cm^{3}min^{-1}

7 **a** 0.035 ms^{-1} **b** 0.035 ms^{-1}

8 8π cm^{3}min^{-1}

9 854 kmh^{-1}

10 $\frac{53}{6}$

11 2 rad s^{-1}

12 **a** $V = h^2 + 8h$ **b** $\frac{4}{15}$ m min^{-1} **c** 0.56 m^{2}min^{-1}

13 $\frac{3\sqrt{10}}{200}$ m min^{-1}

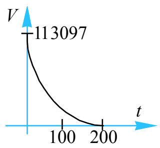
14 $10\sqrt{2}$ cm^{3}s^{-1}

15 0.9 ms^{-1}

16 –3.92 ms^{-1}

17 **a** $x = 30 - 0.15t$ **b** [0, 200] **c i** 1531 cm^{3}s^{-1} **i i** 15.90 cm^{2}s^{-1}

d



18 ~1.24 ms^{-1}

19 ~0.0696 ms^{-1}

20 **a** $y = \sqrt{119 + 20t - 4t^2}$ **b** ~0.516 ms^{-1}

21 **a** 0.095 cms^{-1} **b** 0.6747 cm^{2}s^{-1}

22 **a i** $x = 70t$ **ii** $y = 80t$ **b** $130t$ **c** 130 kmh^{-1}

d 14.66 kmh^{-1}

23 –0.77 ms^{-1}

24 0.40 ms^{-1}

25 3.2 ms^{-1}

26 0.075 m min^{-1}

27 1.26° per sec

28 $\frac{5}{2564} \approx 0.002$ rad per second

29 **a** 9% per second **b** 6% per second

30 0.064

31 8211 per year

32 4% per second

33 –0.25 rad per second

Exercise E.11.1

1 **a** $20x^3$ **b** $48(1+2x)^2$ **c** $\frac{2}{x^3}$ **d** $\frac{2}{(1+x)^3}$

e 2 **f** $\frac{6}{(x-2)^3}$ **g** $\frac{42}{x^8}$ **h** $24(1-2x)$

i $-\frac{1}{x^2}$ **j** $\frac{2(x^2+1)}{(1-x^2)^2}$ **k** $-16\sin 4\theta$ **l** $2\cos x - x\sin x$

m $6x^2\cos x + 6x\sin x - x^3\sin x$ **n** $\frac{1}{x}$ **o** $\frac{10}{(2x+3)^3}$

p $6xe^{2x} + 12x^2e^{2x} + 4x^3e^{2x}$ **q** $\frac{8\sin 4x - 15\cos 4x}{e^x}$

r $2\cos x^2 - 4x^2\sin x^2$ **s** $\frac{-48(x^2+2x^5)}{(4x^3-1)^3}$ **t** $\frac{10}{(x-3)^3}$

2 **a** $6x^2\cos x + (6x - x^3)\sin x$ **b** $2\ln x + 3$

c $\frac{-4}{(x+3)^3}$ **d** $(\cos^2 x - \sin x)e^{\sin x}$

e $\frac{-1}{4(x+1)\sqrt{x+1}}$ **f** $\frac{(4x^2+4x+2)e^{2x}}{(x+1)^3}$

3 $\frac{6\ln x - 5}{x^4}, \frac{n^2\ln x + n\ln x - 2n - 1}{x^{n+2}}$

4 $f'(x) = -\frac{1}{(x+1)^2}, f''(x) = \frac{2}{(x+1)^3}, f^{(iii)}(x) = -\frac{6}{(x+1)^4}, f^{(iv)}(x) = \frac{24}{(x+1)^5}$

$f^{(v)}(x) = -\frac{120}{(x+1)^6}, \ldots\ldots, f^{(n)}(x) = (-1)^n\frac{n!}{(x+1)^{n+1}}$

5 $f(x) = \left(\frac{x+1}{x-1}\right)^n \Rightarrow f''(x) = \frac{4n(n+x)}{(x^2-1)^2}\left(\frac{x+1}{x-1}\right)^n$

7 **a** $a^n e^{ax}$ **b** $\frac{(-1)^n 2^n n!}{(2x+1)^{n+1}}$

c $n = 2k: \quad y^{(n)}(x) = (-1)^k a^{2k}\sin(ax+b), k = 1, 2, \ldots$

$n = 2k-1: \quad y^{(n)}(x) = (-1)^{k+1}a^{2k-1}\cos(ax+b), k = 1, 2, \ldots$

8	**a**	$2+\frac{1}{8\sqrt{2}}$	**b**	$\frac{3+\pi}{2}$

9 −1 [0,1.0768[∪]3.6436,2p]

Exercise E.11.2

1. a 203 b 2.25 c 2.6923

   d 15.91 e 23.9 f 1

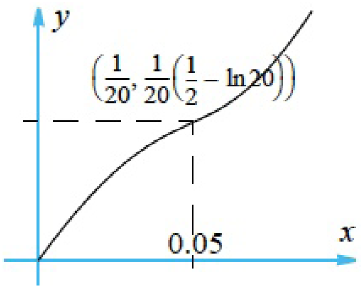
2. a min at (1, 0) max at $\left(-\frac{1}{3},\frac{32}{27}\right)$ b min at $\left(\frac{4}{9},\frac{4}{27}\right)$

   c min at (2, 4), max at (−2, −4) d min at (1, 2), min at (−1, 2)

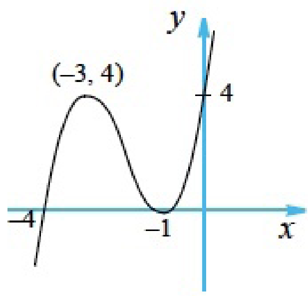
3. min at (1, −3), max at (−3, 29), non-stationary infl (−1, 13)

4. Stationary points: $\left(\frac{\pi}{4},\frac{1}{\sqrt{2}}e^{\left(\frac{\pi}{4}\right)}\right),\left(\frac{5\pi}{4},\frac{1}{\sqrt{2}}e^{\left(\frac{5\pi}{4}\right)}\right)$, inflection points: (0, 1), (π, $-e^{\pi}$), (2π, $e^{2\pi}$).

5.



6.



7. $a = 3$, $b = -20$, $c = 0$.

8. a $4e^{-2}$ b $4.\log_e 4$ c 4

   d 11.25 e 1 f 0

   g

Exercise A.7.1

25. The equation $z + b + i(z - 4) = 0$, where b is a real number, has as its solution a real number. Determine this solution and hence determine the value of b.

26. Express the following in the form $a + bi$ where a and b are real numbers.

a $\dfrac{\cos 2\theta + i\sin 2\theta}{\cos\theta + i\sin\theta}$

b $\dfrac{\cos\theta + i\sin\theta}{\cos 3\theta - i\sin 3\theta}$

27*. Let the complex matrix $A = \begin{bmatrix} \alpha i & 0 \\ 0 & -\beta i \end{bmatrix}$. Find:

a A^2 b A^4

c A^{-1} d A^{4n}, where n is a positive integer.

28*. Find $\dfrac{dy}{d\theta}$ given that $y = \cos\theta + i\sin\theta$.

Show: i $i \cdot \dfrac{d\theta}{dy} = \dfrac{1}{y}$ ii when $\theta = 0, y = 1$.

Hence show that $e^{i\theta} = \cos\theta + i\sin\theta$.

Deduce an expression for $e^{-i\theta}$.

Hence, show: i $\cos x = \dfrac{1}{2}(e^{ix} + e^{-ix})$ ii $\sin x = \dfrac{1}{2i}(e^{ix} - e^{-ix})$.

* questions that are intended for revision. They contain concepts not yet addressed.

Exercise A.7.2

1. Show the following complex numbers on an Argand diagram:

g $\frac{1}{2i}$ h $\frac{2}{1+i}$

3. If $z_1 = 1 + 2i$ and $z_2 = 1 + i$, show each of the following on an Argand diagram:

g $\frac{z_1}{z_2}$ h $\frac{z_2}{z_1}$

4. Find the modulus and argument of:

d $3i$ e $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$ f $\frac{1}{\sqrt{2}}(i+1)$

g 6 h $\left(1 - \frac{1}{2}i\right)^2$

14. Determine the modulus and argument of each of the complex numbers:

a $3 - 4i$ b $\frac{2}{1+i}$ c $\frac{1-i}{1+i}$

15. If $z = 1 + i$ find $Arg(z)$. hence, find $Arg\left(\frac{1}{z^4}\right)$.

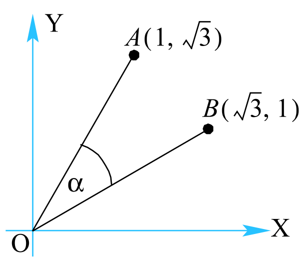
16. Determine the modulus and argument of each of the complex numbers:

a $\cos\theta + i\sin\theta$ b $\sin\theta + i\cos\theta$ c $\cos\theta - i\sin\theta$

17. Find the modulus and argument of:

a $1 + i\tan\alpha$ b $\tan\alpha - i$ c $1 + \cos\theta + i\sin\theta$

18. i Express $\frac{1+\sqrt{3}i}{\sqrt{3}+i}$ in the form $u + vi$.



ii Let α be the angle as shown in the diagram. Use part i to find α, clearly explaining your reason(s).

Hence, find $Arg(z)$ where $z = \left(\frac{1+\sqrt{3}i}{\sqrt{3}+i}\right)^7$.

19. Find:

i the modulus

ii the principal argument of the complex number $1-\cos\theta - i\sin\theta$.

On an Argand diagram, for the case $0 < \theta < \pi$, interpret geometrically the relationship:

$$1-\cos\theta - i\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\left(\cos\left(\frac{\theta-\pi}{2}\right)+i\sin\left(\frac{\theta-\pi}{2}\right)\right).$$

20. If $z = \cos\theta + i\sin\theta$, prove:

a $\frac{2}{1+z} = 1 - i\tan\left(\frac{\theta}{2}\right)$.

b $\frac{1+z}{1-z} = i\cot\left(\frac{\theta}{2}\right)$.

Exercise A.7.4

9. a If $z = cis(\theta)$, show that:

i $z^2 = \cos(2\theta) + i\sin(2\theta)$

ii $z^2 = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$

Hence, show that:

A $\sin 2\theta = 2\sin\theta\cos\theta$ B $\cos 2\theta = \cos^2\theta - \sin^2\theta$ C $\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$

Using the same approach as that in part a, derive the following identities.

a $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

b $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

c $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

10. If $z = cis\theta$, prove: i $z^n + \frac{1}{z^n} = 2\cos n\theta$ ii $z^n - \frac{1}{z^n} = 2i\sin n\theta$

11. If $z = x + iy, y \neq 0$, show that $w = \frac{z}{(1 + z^2)}$, $1 + z^2 \neq 0$, is real, only if $|z| = 1$.

12. Simplify the expression $\frac{1 + i\tan\theta}{1 - i\tan\theta}$. Hence, show that $\left(\frac{1 + i\tan\theta}{1 - i\tan\theta}\right)^k = \frac{1 + i\tan k\theta}{1 - i\tan k\theta}$ where k is a positive integer.

13. Consider the complex number $z = cis\left(\frac{2k\pi}{5}\right)$ for any integer k such that $z \neq 1$.

a Show that $z^n + \frac{1}{z^n} = 2\cos\left(\frac{2nk\pi}{5}\right)$ for any integer n.

b Show that $z^5 = 1$. Hence, or otherwise, show that $1 + z + z^2 + z^3 + z^4 = 0$.

c Find the value of b, given that $\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 = b$.

14. If n is a positive integer, show that $(1 + i)^n + (1 - i)^n = 2^{\frac{n}{2} + 1} \cdot \cos\left(\frac{n\pi}{4}\right)$.

15. Simplify the expression $\frac{(1 + cis(-\theta))^3}{(1 + cis(\theta))^3}$.

16. If $u = cis\theta$ and $v = cis\alpha$ express $\frac{u}{v} + \frac{v}{u}$ in terms of θ and α.

17. a If $cis\alpha = a$ and $cis\beta = b$, prove $\sin(\alpha - \beta) = \frac{b^2 - a^2}{2ab}i$.

b If $(1 + cis\theta)(1 + cis2\theta) = a + bi$, prove $a^2 + b^2 = 16\cos^2\theta\cos^2\left(\frac{\theta}{2}\right)$.

18. If $|z| = 1$ and $Arg(z) = \theta, 0 < \theta < \frac{\pi}{2}$, find: a $\left|\frac{2}{1 - z^2}\right|$ b $\arg\left(\frac{2}{1 - z^2}\right)$

Exercise A.7.5

1. Use the nth root method to solve the following:

   e $z^4 = 81i$ f $z^6 = -64$

8. If 1, w_1 and w_2 are the cube roots of unity, prove:

   a $w_1 = \overline{w_2} = w_2^2$ b $w_1 + w_2 = -1$ c $w_1 w_2 = 1$

9. Given that w is a complex root of the equation $z^5 - 1 = 0$ and is such that it has the smallest positive argument, show that w^2, w^3 and w^4 are the other complex roots.

   a Hence show that $1 + w + w^2 + w^3 + w^4 = 0$.

   b Factorise $z^5 - 1$ into real linear and quadratic factors.

   Hence deduce that: i $2\left(\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right)\right) = -1$

   ii $4\cos\left(\frac{2\pi}{5}\right)\cos\left(\frac{4\pi}{5}\right) = -1$

10. Show that the roots of $(z-1)^6 + (z+1)^6 = 0$ are $\pm i, \pm i\cot\left(\frac{5\pi}{12}\right), \pm i\cot\left(\frac{\pi}{12}\right)$.

Exercise A.9.1

5. Find the solution sets of the following simultaneous equations, solving for x and y.

a
$$\begin{aligned} bx + y &= a \\ ax - y &= b \end{aligned}$$

b
$$\begin{aligned} bx + y &= a \\ ax + y &= b \end{aligned}$$

c
$$\begin{aligned} ax + by &= 1 \\ ax - by &= 1 \end{aligned}$$

d
$$\begin{aligned} ax + y &= ab \\ bx - y &= b^2 \end{aligned}$$

e
$$\begin{aligned} ax + by &= a - b \\ bx + ay &= a - b \end{aligned}$$

f
$$\begin{aligned} ax + y &= b \\ bx + ay &= 2ab - a^3 \end{aligned}$$

Exercise B.6.1

6. Consider the function $f(x) = \frac{2-x}{2+x}$.

a i Find the coordinates of the intercepts with the axes.

ii Determine the equations of the asymptotes of f.

iii Hence, sketch the graph of f.

iv Determine the domain and range of f.

b Find f^{-1} , the inverse function of f.

b Deduce the graph of $(f(x))^2$.

7. Express $\frac{8x-5}{x-3}$ in the form $A + \frac{B}{x-3}$, where A and B are integers.

a Hence, state the equations of the vertical and horizontal asymptotes of the function $f(x) = \frac{8x-5}{x-3}$.

b Sketch the graph of $f(x) = \frac{8x-5}{x-3}$ and use it to determine its range.

8. On different sets of axes, sketch the graphs of $f(x) = 2 + \frac{1}{x}$ and $g(x) = \frac{1}{f(x)}$, stating their domains and ranges.

9. Sketch the graphs of the following functions, clearly labelling all asymptotes.

a $f(x) = 2x + \frac{1}{x}, x \neq 0$ b $g(x) = \frac{1}{2}x + \frac{1}{x^2}, x \neq 0$

c $g(x) = -x + \frac{1}{x}, x \neq 0$ d $f(x) = x - \frac{1}{x}, x \neq 0$

10. Sketch the graphs of the following functions, clearly labelling all asymptotes.

a $h(x) = x^2 + \frac{2}{x}, x \neq 0$ b $f(x) = x^2 + \frac{1}{x^2}, x \neq 0$

c $g(x) = x - \frac{1}{x^2}, x \neq 0$ d $f(x) = x^3 + \frac{3}{x}, x \neq 0$

11. Sketch the graphs of the following functions, clearly labelling all asymptotes.

a $f(x) = x + 3 + \frac{2}{x}, x \neq 0$ b $f(x) = -x + \frac{1}{x} + 2, x \neq 0$

c $g(x) = 2x + \frac{1}{x^2} - 2, x \neq 0$ d $f(x) = \frac{x^2 + 2x - 2}{x}, x \neq 0$

12. a For the function $f(x) = 3 + \frac{1}{1-x} - x$:

i determine all axial intercepts and the coordinates of its stationary points.

ii write down the equation of all the asymptotes.

b Sketch the graph of $y = f(x)$ clearly labelling all the information from part a.

13. Sketch the graphs of: a $f(x) = \frac{x^2 - x - 1}{x - 2}, x \neq 2$. b $g(x) = \frac{(x+2)^2(x-1)}{x^2}, x \neq 0$.

14. Sketch the graphs of the following functions.

a $f(x) = \dfrac{2x - 3}{x^2 - 3x + 2}$ b $y = \dfrac{x^2 + 2x}{x^2 + 4}$ c $y = \dfrac{x^4 + 1}{x^2 + 1}$.

15. Sketch the graph of $f(x) = \dfrac{x + 1}{\sqrt{x - 1}}$, clearly identifying all asymptotes and turning points.

San Diego Tide Data.

Date	Tide 1	Height	Tide 2	Height	Tide 3	Height
1	1:15	4.0	6:56	1.9	13:38	5.2
2	2:47	3.5	7:45	2.4	14:44	5.2
3	4:51	3.4	9:15	2.8	16:04	5.4
4	6:27	3.7	10:56	2.8	17:20	5.8
5	0:44	0.1	7:22	4.2	12:12	2.5
6	1:33	-0.4	8:03	4.6	13:11	2.0
7	2:17	-0.8	8:04	5.1	14:02	1.5
8	2:57	-1.0	9:16	5.5	14:50	1.0
9	3:36	-0.9	9:51	5.8	15:36	0.7
10	4:13	-0.6	10:27	6.0	16:22	0.5
11	4:49	-0.2	11:03	6.0	17:09	0.5
12	5:25	0.5	11:41	5.9	17:59	0.6
13	0:04	5.2	6:01	1.2	12:20	5.7
14	1:00	4.4	6:39	1.9	13:04	5.4
15	2:14	3.8	7:23	2.5	13:57	5.1
16	4:10	3.5	8:32	3.0	15:09	4.9
17	6:24	3.7	10:20	3.2	16:34	4.8
18	0:11	1.0	7:18	4.0	11:52	3.1
19	1:00	0.7	7:47	4.3	12:47	2.7
20	1:36	0.5	8:09	4.6	13:25	2.3
21	2:06	0.3	8:29	4.8	13:57	1.9
22	2:34	0.2	8:50	5.0	14:28	1.6
23	3:00	0.2	9:12	5.3	14:59	1.2
24	3:25	0.2	9:36	5.5	15:31	1.0
25	3:51	0.4	10:01	5.7	16:05	0.8
26	4:17	0.7	10:28	5.8	16:41	0.7
27	4:44	1.0	10:56	5.9	17:21	0.6
28	5:11	1.5	11:27	5.8	6:08	0.7
29	0:18	4.3	5:41	1.9	12:04	5.7
30	1:26	3.8	6:16	2.5	12:53	5.6

Tide 4	Height
20:42	1.6
22:18	1.3
23:42	0.8
18:24	6.3
19:20	6.7
20:10	7.0
20:58	7.1
21:43	7.0
22:29	6.5
23:15	5.9
18:54	0.9
20:00	1.2
21:23	1.3
22:58	1.2
17:46	5.0
18:39	5.3
19:21	5.5
19:57	5.8
20:30	5.9
21:03	5.9
21:36	5.8
22:10	5.6
22:47	5.3
23:28	4.8
19:07	0.8
20:24	0.9